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The formation of optical shocks for paraxial pulses propagating in weakly nonlinear and weakly dispersive materials

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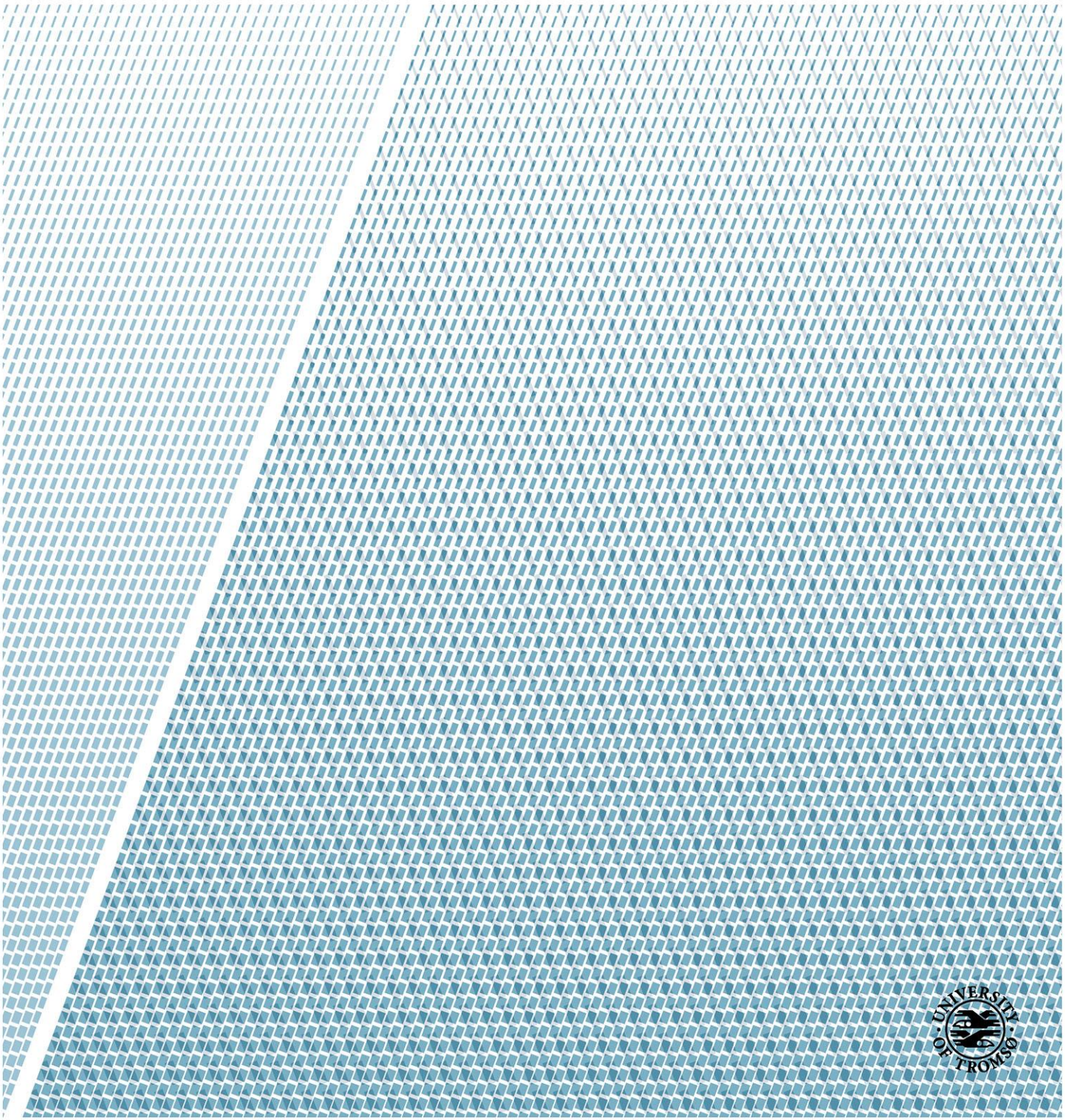

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Abstract

In this thesis leading order asymptotic equations describing the propagation of approximately paraxial pulses in a weakly nonlinear and weakly dispersive medium are derived using the method of multiple scales, and the formation of optical shocks in the nondispersive, purely paraxial case is investigated. How the state of polarization influences the shock have in particularly been looked in to.

Introduction

Linear and nonlinear optics play a fundamental role in todays technology driven society.

From the ability to probe ever deeper into the cosmos using large telescopes filled with the latest adaptive optics systems, to the ubiquitous use of microscopic highly efficient lasers and near perfectly transparent optical fibers in the global Internet, optics and optical technology is front and center.

This technology dream world has been made possible by an ever more refined insight into the way light and matter interact, and the development of computational algorithms to capitalize on this insight.

In this thesis we are first going to use the method of multiple scales [1] to derive leading order asymptotic equations describing the propagation of approximately paraxial pulses in a weakly dispersive and weakly nonlinear medium. This method is a perturbation method that uses the presence of a small dimensionless parameter to reduce a nonlinear problem into an infinite series of linear problems. It uses the presence of breakdown for a direct perturbation expansion, and turns them into solvability conditions. These solvability conditions are enforced by making them into the differential equations called amplitude equations. These equations are a key component in our fast numerical method of solving optical propagation problems.

To build up the competence to derive the asymptotic equations we are going to first use the method of multiple scales on a scalar equation, before using the method on TE vector equations up to second order ϵ, and then to the fourth order in ϵ. This will then give us the skills to be able to use the method to derive amplitude equations from the full Maxwell's equations.

These equations are going to be simplified by looking at the nondispersive, purely paraxial case. By using numerical methods [2],these equations will be investigated. We are particularly interested in the influence of polarization on the formation of optical shocks..

Earlier work that have been done on this subject are for example work done by K. Glasner, M. Kolesik, J. V. Moloney and A. C. Newell [4], where a scalar equation for the electric field as a model for optical shock formation is introduced. Their equation also includes the effect of dispersion, diffraction and nonlinearity. But unlike in this thesis the equation is not derived using a systematic perturbation expansion, and doesn't include effects from nonlinear terms that occur after order ϵ^4. And since it's not a scalar equation can it not be used to investigate the effect of polarization.

Another earlier work is done by A. A. Balakin, A. G. Litvak, V. A. Mironov and S. A. Skobelev [5]. This work also take into account dispersion, diffraction, nonlinearity and polarization. And like [4], and unlike this thesis, are the equations not derived using a systematic perturbation expansion. It only includes a specific dispersion model and doesn't include nonlinear - or

polarization effects that occur at order ϵ^4.

In chapter one we will introduce the Maxwell's equations and simplify them into a scalar equation.

In chapter two we will introduce linear and nonlinear polarization into the scalar equation found in chapter one, and explain the effect which is assumed in this case to be the source of the nonlinear polarization, which is called the Kerr effect.

Chapter three takes into account the presence of temporal dispersion in Maxwell's equations, which makes it impossible to solve them as an initial value problem, and introduces a change of variables to turn it into a boundary value problem.

Chapter four uses the method of multiple scales to derive the leading order asymptotic equations, starting with the TE scalar equation. Then we are moving on to derive the TE vector equations to order ϵ^2, before finally deriving the vector Maxwell equations to the fourth order of ϵ. Starting with the scalar equations and deriving vector equations to second order of ϵ makes it possible for us to develop our skills before deriving our vector Maxwell's equations.

In chapter 5 we look at our perturbation equations in the non-dispersive, purely paraxial case without polarization up to secont order of ϵ, and use numerical methods to investigate the optical shock.

And in chapter 6 we reintroduce the contribution from the fourth order of ϵ without polarization, and use numerical methods to investigate the optical shock to see how big of a contribution this will have.

In chapter 7 we go back to the equations to the second order of ϵ, and introduce polarization, and use numerical methods to investigate how this influences the optical shock.

And then in chapter 8 and 9 we discuss our results and summarize.

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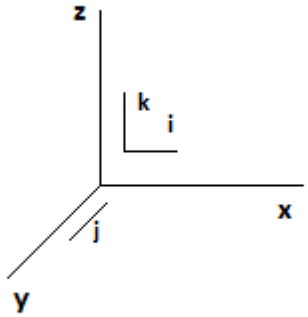

Figure 1: Coordinate system

1 Maxwell's equations

The derivation of the leading order asymptotic equations describing the propagation of approximately paraxial pulses in a weakly nonlinear an weakly dispersive media start with Maxwell's equations.

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \tag{1}$$
$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \partial_t \mathbf{E} + \mu_0 \partial_t \mathbf{P}, \tag{2}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$
$$\nabla \cdot \mathbf{E} = -\frac{1}{\epsilon_0} \nabla \cdot \mathbf{P}. \tag{4}$$

We introduce a cartesian coordinate system where **i**, **j** and **k** unity vectors for three directions whose coordinates are x, y and z. In this first part of the derivation the electric field and the polarization will only have contributions in the y-direction, while the magnetic field will have a contributions in the x- and z-direction. These are called transverse electric fields(TE).

$$\mathbf{B} = B_x(x, z, t)\mathbf{i} + B_z(x, z, t)\mathbf{k}, \tag{5}$$
$$\mathbf{E} = E(x, z, t)\mathbf{j}, \tag{6}$$
$$\mathbf{P} = P(x, z, t)\mathbf{j}. \tag{7}$$

Start by inserting (5)-(7) into the Maxwell's equations (1)-(4).

For equation (1) will this give:

$$\nabla \times \mathbf{E} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & E & 0 \end{vmatrix} = -\frac{\partial E}{\partial z}\mathbf{i} + 0 + \frac{\partial E}{\partial x}\mathbf{k}, \tag{8}$$

$$\implies -\partial_z E\mathbf{i} + \partial_x E\mathbf{k} + \partial_t B_x\mathbf{i} + \partial_t B_z\mathbf{k} = 0. \tag{9}$$

Which implies

$$-\partial_z E + \partial_t B_x = 0, \tag{10}$$

$$\partial_x E + \partial_t B_z = 0. \tag{11}$$

And for equation (2) of Maxwell's equation inserting (5)-(7) gives:

$$\nabla \times \mathbf{B} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ B_x & 0 & B_z \end{vmatrix} \tag{12}$$

$$= 0 + (\partial_z B_x - \partial_x B_z)\mathbf{j} + 0. \tag{13}$$

Using (13) on equation (2) turns into

$$(\partial_z B_x - \partial_x B_z)\mathbf{j} - \epsilon_0\mu_0\partial_t E\mathbf{j} = \mu_0\partial_t P\mathbf{j}, \tag{14}$$

Thus

$$\partial_z B_x - \partial x B_z - \epsilon_0\mu_0\partial_t E = \mu_0\partial_t P. \tag{15}$$

Inserting (5) - (7) into equation (3) we get

$$\nabla \cdot \mathbf{B} = \partial_x B_x + \partial_z B_z = 0, \tag{16}$$

and thus we have

$$\partial_x B_x + \partial_z B_z = 0. \tag{17}$$

The last of Maxwell's equations is automatically satisfied because

$$\nabla \cdot \mathbf{E} = \partial_y E\mathbf{j} = 0, \tag{18}$$

and

$$\nabla \cdot \mathbf{P} = \partial_y P\mathbf{j} = 0. \tag{19}$$

The system (1) - (4) has thus been simplified into

$$-\partial_z E + \partial_t B_x = 0, \tag{20}$$
$$\partial_x E + \partial_t B_z = 0, \tag{21}$$
$$\partial_z B_x - \partial_x B_z - \epsilon_0\mu_0\partial_t E = \mu_0\partial_t P, \tag{22}$$
$$\partial_x B_x + \partial_z B_z = 0. \tag{23}$$

We will use equations (20)- (22) to eliminate B_x and B_y and will end up with equations for E and P only. This is done by taking cross derivatives of these equations such that equations (22) and (23) can be inserted into equations (24).

The derivative of equation(22) with respect to t is

$$\partial_{zt} B_x - \partial_{xt} B_z - \epsilon_0\mu_0\partial_{tt} E = \mu_0\partial_{tt} P. \tag{24}$$

The derivative of equation (20) with respect to z is

$$-\partial_{zz} E + \partial_{tz} B_x = 0,$$

or solving with respect to B_x

$$\partial_{zt} B_x = \partial_{zz} E. \tag{25}$$

And the derivative of equation (21) with respect to x is

$$\partial_{xx} E + \partial_{xt} B_z = 0,$$

which give us

$$\partial_{tx} B_z = -\partial_{xx} E. \tag{26}$$

Inserting equations (20) and (21) into equation (22) gives the following equation involving only E and P.

$$\partial_{zz} E + \partial_{xx} E - \epsilon_0\mu_0\partial_{tt} E = \mu_0\partial_{tt} P. \tag{27}$$

2 Polarization

The polarization is generally a sum of terms that are linear in **E** and that are nonlinear in **E**,

$$P = P_L + P_{NL}, \tag{28}$$

2.1 The Kerr effect

Generally the nonlinear polarization can come from any source, but when doing concrete calculations will we for simplicity assume that the nonlinear polarization comes from the Kerr effect [3]. This is a phenomenon where the refractive index changes because of a sufficiently strong electrical field, and arises because of the off-resonance electronic response of the atoms and molecules that are exposed to this field. For materials with inversion symmetry the nonlinear polarization will be given by $P_{NL} = \epsilon_0 \eta E^3$, where η is the Kerr coefficient[3].

2.2 Linear Polarization

The linear polarization has the general form

$$P_L = \epsilon_0 \int_{-\infty}^{t} dt' \chi(t - t') E(t'). \tag{29}$$

This means that the polarization at time t is dependent on the electric field at all times before t. This is called temporal dispersion. The presence of this in Maxwell's equations makes it impossible to solve them as a standard initial value problem.

A more convenient representation of the temporal dispersion is found by rewriting the linear polarization using the convolution theorem.

$$P_L = \epsilon_0 \int_{-\infty}^{\infty} d\omega \hat{\chi}(\omega) \hat{E}(\omega) e^{-i\omega t}. \tag{30}$$

Using the Taylor expansion of $\chi(\omega)$ around $\omega = 0$ we have

$$
\begin{aligned}
P_L &= \epsilon_0 \int_{-\infty}^{\infty} d\omega \sum_{n=0}^{\infty} \frac{\hat{\chi}^{(n)}(0)}{n!} \omega^n \hat{E}(\omega) e^{-i\omega t} \\
&= \epsilon_0 \sum_{n=0}^{\infty} \frac{\hat{\chi}^{(n)}(0)}{n!} \int_{-\infty}^{\infty} d\omega \cdot \omega^n \hat{E}(\omega) e^{-i\omega t} \\
&= \epsilon_0 \sum_{n=0}^{\infty} \frac{\hat{\chi}^{(n)}(0)}{n!} \int_{-\infty}^{\infty} d\omega (i\partial_t)^n \hat{E}(\omega) e^{-i\omega t} \\
&= \epsilon_0 \sum_{n=0}^{\infty} \frac{\hat{\chi}^{(n)}(0)}{n!} (i\partial_t)^n \int_{-\infty}^{\infty} d\omega \hat{E}(\omega) e^{-i\omega t}.
\end{aligned}
\tag{31}
$$

Where

$$
\int_{-\infty}^{\infty} d\omega \hat{E}(\omega) e^{-i\omega t} = E(t), \tag{32}
$$

Thus

$$
P_L = \epsilon_0 \hat{\chi}(i\partial t) E(t). \tag{33}
$$

Inserting equation (28) into equation (27) will give

$$
\partial_{tt} E - c^2 \partial_{zz} E = c^2 \partial_{xx} E - \partial_{tt} \hat{\chi}(i\partial_t) E - \epsilon_0 \eta \partial_{tt} E^3. \tag{34}
$$

Applications of equation (34) usually starts by doing some sort of scaling. This means that one choose some relevant scales for space, time and the electrical field E such as to render the equation dimensionless. In this thesis we will only consider scales where the terms representing diffraction, dispersion and nonlinearity are small and of the same order. For our calculations we will introduce a formal perturbation parameter, ϵ^2, in the dispersive and nonlinear terms and use the space scale $x = \epsilon x_1$

$$
\partial_{tt} E - c^2 \partial_{zz} E = c^2 \partial_{xx} E - \epsilon^2 \partial_{tt} \hat{\chi}(i\partial_t) E - \epsilon^2 \epsilon_0 \eta \partial_{tt} E^3. \tag{35}
$$

where $\epsilon << 1$

3 Change of variables

Because of the presence of temporal dispersion in Maxwell's equations it will be impossible to solve the equations as a standard initial value problem. We will avoid this problem by rather solving the equations as a boundary value problem. This is the way experiments in nonlinear optics are usually done, where a laser pulse is launched into a medium through a boundary.This is thus natural way of solving problems in optics. The way to change the initial value problem into a boundary value problem is to use a change of variables.

For the lowest order of ϵ equation (35) will look like this:

$$\partial_{tt}E - c^2\partial_{zz}E = 0. \tag{36}$$

This is a homogeneous one- dimensional wave equation, and will have a general solution that is the sum of waves that are propagating both left and right along the z-axis, $E(z-ct)$ and $E(z+ct)$.

We introduce the change of variables:

$$\theta = z - ct, \tag{37}$$

$$\tau = z. \tag{38}$$

Using the chain rule to find the partial derivatives will give:

$$\partial_z = \frac{\partial\tau}{\partial z}\frac{\partial}{\partial\tau} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} = \partial_\tau + \partial_\theta, \tag{39}$$

$$\partial_t = \frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau} + \frac{\partial\theta}{\partial t}\frac{\partial}{\partial\theta} = 0 + (-c)\partial_\theta = -c\partial_\theta. \tag{40}$$

From these equations we get

$$\partial_{zz} = \partial_{\tau\tau} + 2\partial_{\tau\theta} + \partial_{\theta\theta}, \tag{41}$$

and

$$\partial_{tt} = c^2\partial_{\theta\theta}. \tag{42}$$

Inserting these change of variables into equation (35) give us the equation

$$2\partial_{\theta\tau}E = -\partial_{xx}E + \epsilon^2\partial_{\theta\theta}\hat{\chi}(-ic\partial_\theta)E + \epsilon^2\eta\partial_{\theta\theta}E^3 - \partial_{\tau\tau}E. \tag{43}$$

Because of the fact that $e^{2i\tau}\cdot e^{-i\theta} = e^{2iz}\cdot e^{-i(z-ct)} = e^{i(2z-z+ct)} = e^{i(z+ct)}$, we still haven't made any assumptions about the solution, equation (43) and (34) are equivalent.

Note that through the change of variables from equation (37) and (38), the line $(\theta, 0)$ in the (θ, τ) plane corresponds to the line $(0, t)$ in the (z, t) plane. This means that the optical propagation problem is now a boundary value problem. For this type of problem it is necessary to make sure that the pulse traveling to the right into the medium does not create a significant pulse that is traveling to the left. If that happens the problem will not make sense mathematically. This is because the pulse traveling to the left will eventually hit the boundary at z=0. When that happens, the field at z=0 will not only consist of the initial pulse, but also have contributions from the left traveling pulse. The left traveling pulse is unknown until the equation is solved, and that means that the propagation problem will not be well posed mathematically as a boundary value problem if this left traveling pulse is of any significant size. We will therefore look for solutions of the form $E = E(z - ct, \epsilon z)$ that is a small perturbation on a purely right traveling wave.

4 The Method of multiple scales

4.1 TE scalar equation

We will now use the multiple scale method [1] to find the approximation solution to equation (43). Introduce the function $e(\theta, x_1, \tau_1, \tau_2, ...)$ where $x_1 = \epsilon x$ and $\tau_j = \epsilon^j \tau$ and make the expansions:

$$\partial_\tau = \epsilon \partial_{\tau_1} + \epsilon^2 \partial_{\tau_2} + ..., \tag{44}$$

$$\partial_x = \epsilon \partial_{x_1}, \tag{45}$$

$$e = e_0 + \epsilon e_1 + \epsilon^2 e_2 + ... \tag{46}$$

Inserting the expansions (44)-(46) into equation (43) gives us the equation

$$\begin{aligned} &2\partial_\theta(\epsilon\partial_{\theta_1} e_0 + \epsilon^2 \partial_{\theta_1} e_1 + \epsilon^2 \partial_{\theta_2} e_0) = -\epsilon^2 \partial_{x_1 x_1} e_0 \\ &+\epsilon^2 \epsilon_0 \partial_{\theta\theta} \hat{\chi}(-ic\partial_\theta) e_0 + \epsilon^2 \epsilon_0 \eta \partial_{\theta\theta} e_0^3 - \epsilon^2 \partial_{\tau_1 \tau_1} e_0. \end{aligned} \tag{47}$$

And this equation gives us the following perturbation hierarchy to second order in ϵ

$$\epsilon^1 : 2\partial_{\tau_1 \theta} e_0 = 0, \tag{48}$$

$$\begin{aligned} \epsilon^2 : 2\partial_{\theta\tau_1} e_1 = &-2\partial_{\theta\tau_2} e_0 - \partial_{x_1 x_1} e_0 + \epsilon_0 \partial_{\theta\theta} \hat{\chi}(-ic\partial_\theta) e_0 \\ &+ \epsilon_0 \eta \partial_{\theta\theta} e_0^3 - \partial_{\tau_1 \tau_1} e_0. \end{aligned} \tag{49}$$

The general solution to (48) is

$$e_0 = e_0(x_1, \theta, \tau_2, ..). \tag{50}$$

Where we have disregarded an arbitrary function of the form

$$\alpha = \alpha(x_1, \tau_1, \tau_2 ...). \tag{51}$$

The solution (51) implies that

$$\partial_{\tau_1 \tau_1} e_0 = 0. \tag{52}$$

Since the right hand side of (49) does not depend on τ_1, we will get a secular growth and breakdown of our perturbation expansion (46) when

$$\tau_1 \sim \frac{1}{\epsilon}. \tag{53}$$

In order to avoid this we must impose the solvability condition

$$-2\partial_{\tau_2\theta}e_0 - \partial_{x_1x_1}e_0 + \epsilon_0\partial_{\theta\theta}\hat{\chi}(-ic\partial_\theta)e_0 + \epsilon_0\eta\partial_{\theta\theta}e_0^3 = 0. \tag{54}$$

Using (52) our order ϵ^2 equation simplifies into

$$2\partial_{\theta\tau_1}e_1 = 0. \tag{55}$$

According to the rules of the game [2], we choose the special solution

$$e_1 = 0. \tag{56}$$

This will give us the equation

$$2\partial_{\tau_2\theta}e_0 = -\partial_{x_1x_1}e_0 + \epsilon_0\partial_{\theta\theta}\hat{\chi}(-ic\partial_\theta)e_0 + \epsilon_0\eta\partial_{\theta\theta}e_0^3. \tag{57}$$

Define

$$E_0(\theta, x, \tau) = e_0(\theta, x, \tau_2, ..)|_{x_1=\epsilon x, \tau_j=\epsilon^j\tau}. \tag{58}$$

Then multiplying (57) with ϵ^2, using (58) we get the amplitude equation

$$2\partial_{\theta\tau}E_0 = -\partial_{xx}E_0 + \epsilon^2\epsilon_0\partial_{\theta\theta}\hat{\chi}(-ic\partial_\theta)E_0 + \epsilon^2\epsilon_0\eta\partial_{\theta\theta}E_0^3. \tag{59}$$

4.2 TE vector equations order ϵ^2

We are now moving on with building up a competence to do the derivations on chapter 4.4. This is why we are using the change of (37) and (38) on Maxwell's equations instead of the scalar equation (27). So, doing a change of variables from equations (37) to (42) on Maxwell's equations will give us the system of equations

$$\partial_\tau E + \partial_\theta E + c\partial_\theta B_x = 0, \tag{60}$$

$$\partial_x E - c\partial_\theta B_z = 0, \tag{61}$$

$$\partial_\tau B_x + \partial_\theta B_x - \partial_x B_z + \frac{1}{c}\partial_\theta E = -\mu_0 c\partial_\theta P, \tag{62}$$

$$\partial_x B_x + \partial_\tau B_z + \partial_\theta B_z = 0. \tag{63}$$

We solve this system of equations by the multiple scale method, and start by introducing the functions

$$e = e(\theta, x_1, \tau_1, \tau_2, ..), \tag{64}$$
$$b_x = b_x(\theta, x_1, \tau_1, \tau_2, ..), \tag{65}$$
$$b_z = b_z(\theta, x_1, \tau_1, \tau_2, ..), \tag{66}$$
$$p = p(\theta, x_1, \tau_1\tau_2, ...). \tag{67}$$

Introduce the expansions up to the second order of ϵ:

$$e = e_0 + \epsilon e_1 + \epsilon^2 e_2 + ..., \tag{68}$$
$$b_x = b_{x0} + \epsilon b_{x_1} + \epsilon^2 b_{x_2}, \tag{69}$$
$$b_z = b_{z0} + \epsilon b_{z_1} + \epsilon^2 b_{z_2}, \tag{70}$$
$$p = \epsilon p_1 + \epsilon^2 p_2, \tag{71}$$
$$\partial_\tau = \epsilon \partial_{\tau_1} + \epsilon^2 \partial_{\tau_2}, \tag{72}$$
$$\partial_x = \epsilon \partial_{x_1}, \tag{73}$$

where $\tau^j = \epsilon^j \tau$. When we insert the expansions (68)-(73) into the equations (60)-(63) we end up with four perturbation hierarchies. The perturbation hierarchy for equation (60) is

$$\epsilon^0 : \partial_\theta e_0 + c\partial_\theta b_{x_0} = 0, \tag{74}$$
$$\epsilon^1 : \partial_\theta e_1 + c\partial_\theta b_{x_1} = -\partial_{\tau_1} e_0, \tag{75}$$
$$\epsilon^2 : \partial_\theta e_2 + c\partial_\theta b_{x_2} = -\partial_{\tau_1} e_1 - \partial_{\tau_2} e_0. \tag{76}$$

The perturbation hierarchy for equation (61) is

$$\epsilon^0 : c\partial_\theta b_{z_0} = 0, \tag{77}$$
$$\epsilon^1 : c\partial_\theta b_{z_1} = \partial_{x_1} e_0, \tag{78}$$
$$\epsilon^3 : c\partial_\theta b_{z_2} = \partial_{x_1} e_1. \tag{79}$$

The perturbation hierarchy for equation (62) is

$$\epsilon^0 : \partial_\theta b_{x_0} + \frac{1}{c}\partial_\theta e_0 = 0, \tag{80}$$
$$\epsilon^1 : \partial_\theta b_{x_1} + \frac{1}{c}\partial_\theta e_1 = -\partial_{\tau_1} b_{x_0} + \partial_{x_1} b_{z_0} - \mu_0 c \partial_\theta p_1, \tag{81}$$
$$\epsilon^2 : \partial_\theta b_{x_2} + \frac{1}{c}\partial_\theta e_2 = -\partial_{\tau_1} b_{x_1} - \partial_{\tau_2} b_{x_0} + \partial_{x_1} b_{z_1} - \mu_0 c \partial_\theta p_2. \tag{82}$$

And the perturbation hierarchy for equation (63) is

$$\epsilon^0 : \partial_\theta b_{z_0} = 0, \tag{83}$$
$$\epsilon^1 : \partial_\theta b_{z_1} = -\partial_{x_1} b_{x_0} - \partial_{\tau_1} b_{z_0}, \tag{84}$$
$$\epsilon^2 : \partial_\theta b_{z_2} = -\partial_{x_1} b_{x_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}. \tag{85}$$

The solution to these system of equations will be found separately order by order in ϵ. At order ϵ^0 we have equations (74), (77), (80) and (83) which we will write as the system

$$\partial_\theta e_0 + c\partial_\theta b_{x_0} = 0, \tag{86}$$
$$c\partial_\theta b_{z_0} = 0, \tag{87}$$
$$\partial_\theta b_{x_0} + \frac{1}{c}\partial_\theta e_0 = 0, \tag{88}$$
$$\partial_\theta b_{z_0} = 0. \tag{89}$$

Equation (87) has the general solution

$$b_{z_0} = \alpha(x_1, \tau_1, \tau_2, ...). \tag{90}$$

Since these solutions doesn't depend on θ we will disregard them and choose

$$b_{z_0} = 0. \tag{91}$$

Equation (86) has the general solution

$$e_0 + cb_{x_0} = \beta(x_1, \tau_1, \tau_2, ...). \tag{92}$$

As before we disregard β because it doesn't depend on θ, and we get the following expression for b_{x_0}

$$b_{x_0} = -\frac{1}{c}e_0. \tag{93}$$

We observe that (88) and (89) are automatically satisfied, which means that there are no solvability conditions at this order.

The equations from ϵ^1, equations (75), (78), (81) and (84) are written as the system:

$$\partial_\theta e_1 + c\partial_\theta b_{x_1} = -\partial_{\tau_1} e_0, \tag{94}$$
$$c\partial_\theta b_{z_1} = \partial_{x_1} e_0, \tag{95}$$
$$\partial_\theta b_{x_1} + \frac{1}{c}\partial_\theta e_1 = -\partial_{\tau_1} b_{x_0} + \partial_{x_1} b_{z_0} - \mu_0 c\partial_\theta p_1, \tag{96}$$
$$\partial_\theta b_{z_1} = -\partial_{x_1} b_{x_0} - \partial_{\tau_1} b_{z_0}. \tag{97}$$

Starting with equation (96) and (94). In order have a solution we have to impose the solvability condition

$$-\partial_{\tau_1} e_0 = -c\partial_{\tau_1} b_{x_0} + c\partial_{x_1} b_{z_0} - \mu_0 c^2 \partial_\theta p_1. \tag{98}$$

And for equation (95) and (97) we get the solvability condition

$$\partial_{x_1} e_0 = -c\partial_{x_1} b_{x_0} - c\partial_{\tau_1} b_{z_0}. \tag{99}$$

When the solvability conditions are imposed, the equations (86) and (94) becomes under-determined, and it is possible to choose the special solution

$$e_1 = 0. \tag{100}$$

Which give us the equation

$$c\partial_\theta b_{x_1} = -\partial_{\tau_1} e_0. \tag{101}$$

And the solvability condition for (87) and (89) gives us the equation

$$\partial_\theta b_{z_1} = \frac{1}{c}\partial_{x_1} e_0. \tag{102}$$

The equations from ϵ^2, (76), (79), (82) and (85), can be written as the system:

$$\partial_\theta e_2 + c\partial_\theta b_{x_2} = -\partial_{\tau_1} e_1 - \partial_{\tau_2} e_0, \tag{103}$$

$$c\partial_\theta b_{z_2} = \partial_{x_1} e_1, \tag{104}$$

$$\partial_\theta b_{x_2} + \frac{1}{c}\partial_\theta e_2 = -\partial_{\tau_1} b_{x_1} - \partial_{\tau_2} b_{x_0} + \partial_{x_1} b_{z_1} - \mu_0 c \partial_\theta p_2, \tag{105}$$

$$\partial_\theta b_{z_2} = -\partial_{x_1} b_{x_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}. \tag{106}$$

For the pair of equations (103) and (105) we get the solvability condition

$$-\partial_{\tau_1} e_1 - \partial_{\tau_2} e_0 = -c\partial_{\tau_1} b_{x_1} - c\partial_{\tau_2} b_{x_0} + c\partial_{x_1} b_{z_1} - \mu_0 c^2 \partial_\theta p_2. \tag{107}$$

And the solvability condition from (104) and (106) is

$$\partial_{x_1} e_1 = -c\partial_{x_1} b_{x_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}. \tag{108}$$

When the solvability condition (107) is imposed, (103) will become underdetermined, and it's possible to choose without loss of generality the value

$$e_2 = 0. \tag{109}$$

By choosing $e_2 = 0$ will we get

$$c\partial_\theta b_{x_2} = -\partial_{\tau_2} e_0. \tag{110}$$

So far we have found equations

$$\begin{aligned}
&b_{z_0} = 0,\\
&b_{x_0} = -\frac{1}{c} e_0,\\
&c\partial_\theta b_{z_1} = \partial_{x_1} e_0,\\
&c\partial_\theta b_{x_1} = -\partial_{\tau_1} e_0,\\
&e_1 = 0,\\
&e_2 = 0.
\end{aligned}$$

Inserting these into the solvability condition (98) give the equation

$$2\partial_{\tau_1} e_0 = \mu_0 c^2 \partial_\theta p_1, \tag{111}$$

and inserting them into equation (107) give the equation

$$2\partial_{\theta\tau_1} e_0 = -\partial_{\tau_1\tau_1} e_0 + \partial_{x_1 x_1} e_0 + \mu_0 c^2 \partial_{\theta\theta} p_2. \tag{112}$$

Equation (99) becomes

$$\partial_{x_1} e_0 = -c\partial_{x_1} b_{x_0}, \tag{113}$$

which is automatically satisfied. We consider the special case when

$$P = \epsilon^2 \left(\epsilon_0 \hat{\chi}(-ic\partial_\theta) e_0 + \epsilon_0 \eta e_0^3\right), \tag{114}$$

$$\implies p_1 = 0, \tag{115}$$

$$p_2 = \left(\epsilon_0 \hat{\chi}(-ic\partial_\theta) e_0 + \epsilon_0 \eta e_0^3\right). \tag{116}$$

From this we get

$$2\partial_{\theta\tau_1} e_0 = 0, \tag{117}$$
$$2\partial_{\theta\tau_1} e_0 = -\partial_{x_1x_1} e_0 + \mu_0 c^2 \epsilon_0 \partial_{\theta\theta} \hat{\chi}(-ic\partial_\theta) e_0 + \mu_0 \epsilon_0 c^2 \eta \partial_{\theta\theta} (e_0)^3. \tag{118}$$

Introduce

$$E_0(\theta, x, \tau) = e_0(\theta.x_1, \tau_1, \tau_2, ...)|_{x_1=\epsilon_x, \tau_j=\epsilon^j\tau}. \tag{119}$$

Multiplying (117) by ϵ and (118) by ϵ^2, adding and using (119) and using the expansions

$$\partial_x = \epsilon\partial_{x_1}, \tag{120}$$
$$\partial_\tau = \epsilon\partial_{\tau_1} + \epsilon^2\partial_{\tau_2}, \tag{121}$$

we get

$$2\partial_{\theta\tau} E_0 = -\partial_{xx} E_0 + \epsilon^2 \epsilon_0 \partial_{\theta\theta}(-ic\partial_\theta) E_0 + \epsilon^2 \epsilon_0 \eta \partial_{\theta\theta} E_0^3. \tag{122}$$

Which is the same equation as the one we got by applying the multiple scale method to the scalar equation (43).

4.3 TE vector equations order ϵ^4

Moving on with the last example before the derivation of our equations. This time are we starting with equations (60)-(63)

$$\begin{aligned}
&\partial_\tau E + \partial_\theta E + c\partial_\theta B_x = 0,\\
&\partial_x E - c\partial_\theta B_z = 0,\\
&\partial_\tau B_x + \partial_\theta B_x - \partial_x B_z + \frac{1}{c}\partial_\theta E = -\mu_0 c \partial_\theta P,\\
&\partial_x B_x + \partial_\tau B_z + \partial_\theta B_z = 0.
\end{aligned}$$

Introducing the functions (64)-(67)

$$\begin{aligned}
e &= e(\theta, x_1, \tau_1, \tau_2, ..),\\
b_x &= b_x(\theta, x_1, \tau_1, \tau_2, ..),\\
b_z &= b_z(\theta, x_1, \tau_1, \tau_2, ..),\\
p &= p(\theta, x_1, \tau_1\tau_2, ...).
\end{aligned}$$

And introducing the expansions, that this time is going to go up to the fourth order of ϵ

$$e = e_0 + \epsilon e_1 + \epsilon^2 e_2 + \epsilon^3 e_3 + \epsilon^4 e_4, \tag{123}$$
$$b_x = b_{x_0} + \epsilon b_{x_1} + \epsilon^2 b_{x_2} + \epsilon^3 b_{x_3} + \epsilon^4 b_{x_4}, \tag{124}$$
$$b_z = b_{z_0} + \epsilon b_{z_1} + \epsilon^2 b_{z_2} + \epsilon^3 b_{z_3} + \epsilon^4 b_{z_4} + ..., \tag{125}$$
$$p = \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + \epsilon^4 p_4 + ..., \tag{126}$$
$$\partial_\tau = \epsilon \partial_{\tau_1} + \epsilon^2 \partial_{\tau_2} + \epsilon^3 \partial_{\tau_3} + \epsilon^4 \partial_{\tau_4} + ..., \tag{127}$$
$$\partial_x = \epsilon \partial_{x_1}. \tag{128}$$

Inserting the expansions (123)-(128) into the equations (60)-(63) and expanding will give us the perturbation hierarchy to order four in ϵ

$$\epsilon^0 : \partial_\theta e_0 + c\partial_\theta b_{x_0} = 0, \tag{129}$$
$$\epsilon^1 : \partial_\theta e_1 + c\partial_\theta b_{x_1} = -\partial_{\tau_1} e_0, \tag{130}$$
$$\epsilon^2 : \partial_\theta e_2 + c\partial_\theta b_{x_2} = -\partial_{\tau_1} e_1 - \partial_{\tau_2} e_0, \tag{131}$$
$$\epsilon^3 : \partial_\theta e_3 + c\partial_\theta b_{x_3} = -\partial_{\tau_1} e_2 - \partial_{\tau_2} e_1 - \partial_{\tau_3} e_0, \tag{132}$$
$$\epsilon^4 : \partial_\theta e_4 + c\partial_\theta b_{x_4} = -\partial_{\tau_1} e_3 - \partial_{\tau_2} e_2 - \partial_{\tau_3} e_1 - \partial_{\tau_4} e_0. \tag{133}$$

The perturbation hierarchy for equation (61) is

$$\epsilon^0 : c\partial_\theta b_{z_0} = 0, \tag{134}$$
$$\epsilon^1 : c\partial_\theta b_{z_1} = \partial_{x_1} e_0, \tag{135}$$
$$\epsilon^2 : c\partial_\theta b_{z_2} = \partial_{x_1} e_1, \tag{136}$$
$$\epsilon^3 : c\partial_\theta b_{z_3} = \partial_{x_1} e_2, \tag{137}$$
$$\epsilon^4 : c\partial_\theta b_{z_4} = \partial_{x_1} e_3. \tag{138}$$

The perturbation hierarchy for equation (62) is

$$\epsilon^0 : \partial_\theta b_{x_0} + \frac{1}{c}\partial_\theta e_0 = 0, \tag{139}$$
$$\epsilon^1 : \partial_\theta b_{x_1} + \frac{1}{c}\partial_\theta e_1 = -\mu_0 c\partial_\theta p_1 - \partial_{\tau_1} b_{x_0} + \partial_{x_1} b_{z_0}, \tag{140}$$
$$\epsilon^2 : \partial_\theta b_{x_2} + \frac{1}{c}\partial_\theta e_2 = -\mu_0 c\partial_\theta p_2 - \partial_{\tau_2} b_{x_0} - \partial_{\tau_1} b_{x_1} + \partial_{x_1} b_{z_1}, \tag{141}$$
$$\epsilon^3 : \partial_\theta b_{x_3} + \frac{1}{c}\partial_\theta e_3 = -\mu_0 c\partial_\theta p_3 - \partial_{\tau_1} b_{x_2} - \partial\tau_2 b_{x_1} - \partial\tau_3 b_{x_0} + \partial_{x_1} b_{z_2}, \tag{142}$$
$$\epsilon^4 : \partial_\theta b_{x_4} + \frac{1}{c}\partial_\theta e_4 = -\mu_0 c\partial_\theta p_4 - \partial_{\tau_1} b_{x_3} - \partial_{\tau_2} b_{x_2} - \partial_{\tau_3} b_{x_1} - \partial_{\tau_4} b_{x_0} + \partial_{x_1} b_{z_3}. \tag{143}$$

And the perturbation hierarchy for equation (63) is

$$\epsilon^0 : \partial_\theta b_{z_0} = 0, \tag{144}$$
$$\epsilon^1 : \partial_\theta b_{z_1} = -\partial_{x_1} b_{x_0} - \partial_{\tau_1} b_{z_0}, \tag{145}$$
$$\epsilon^2 : \partial_\theta b_{z_2} = -\partial_{x_1} b_{x_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}, \tag{146}$$
$$\epsilon^3 : \partial_\theta b_{z_3} = -\partial_{x_1} b_{x_2} - \partial_{\tau_1} b_{z_2} - \partial_{\tau_2} b_{z_1} - \partial_{\tau_3} b_{z_0}, \tag{147}$$
$$\epsilon^4 : \partial_\theta b_{z_4} = -\partial_{x_1} b_{x_3} - \partial_{\tau_1} b_{z_3} - \partial_{\tau_2} b_{z_2} - \partial_{\tau_3} b_{z_1} - \partial_{\tau_4} b_{z_4}. \tag{148}$$

The solutions will again be found separately order by order in ϵ. The equations for ϵ^0, equations (129), (134), (139) and (144), which we write as the system

$$\partial_\theta e_0 + c\partial_\theta b_{x_0} = 0, \tag{149}$$
$$c\partial_\theta b_{z_0} = 0, \tag{150}$$
$$\partial_\theta b_{x_0} + \frac{1}{c}\partial_\theta e_0 = 0, \tag{151}$$
$$\partial_\theta b_{z_0} = 0. \tag{152}$$

It's easy to see that equations (150) and (152) are equivalent, which will give us

$$\partial_\theta b_{z_0} = 0. \tag{153}$$

The general solution to (153) is

$$b_{z_0} = \beta(x_1, \tau_1, \tau_2, ...). \tag{154}$$

Equation (154) doesn't depend on θ, and therefore will we disregard it and choose

$$b_{z_0} = 0. \tag{155}$$

It's also easy to see that equation (149) equals to c multiplied by (151). This will in the same way give us the general solution

$$cb_{x_0} + e_0 = \alpha(x_1, \tau_1, \tau_2, ...). \tag{156}$$

This will also be disregarded to give the equation

$$b_{x_0} = -\frac{1}{c} e_0. \tag{157}$$

Moving on to the equations for ϵ^1, which are equations (130), (135), (140) and (145).

$$\partial_\theta e_1 + c\partial_\theta b_{x_1} = -\partial_{\tau_1} e_0, \tag{158}$$
$$c\partial_\theta b_{z_1} = \partial_{x_1} e_0, \tag{159}$$
$$\partial_\theta b_{x_1} + \frac{1}{c}\partial_\theta e_1 = -\partial_{\tau_1} b_{x_0} - \partial_{x_1} b_{z_0} - \mu_0 c \partial_\theta p_1, \tag{160}$$
$$\partial_\theta b_{z_1} = -\partial_{x_1} b_{x_0} - \partial_{\tau_1} b_{z_0}. \tag{161}$$

In order for (158) and (160) to have a solution, we have to impose the solvability condition

$$-\partial_{\tau_1} e_0 = -c\partial_{\tau_1} b_{x_0} - c\partial_{x_1} b_{z_0} - \mu_0 c^2 \partial_\theta p_1. \tag{162}$$

And in order for (159) and (161) to have a solution, we have to impose the solvability condition

$$\partial_{x_1} e_0 = -c\partial_{x_1} b_{x_0} - c\partial_{\tau_1} b_{z_0}. \tag{163}$$

When the solution condition (162) is imposed, equation (158) becomes multivalued. This makes it possible to choose without loss of generality

$$e_1 = 0. \tag{164}$$

When $e_1 = 0$,equation (158) will become

$$\partial_\theta b_{x_1} = -\frac{1}{c}\partial_{\tau_1} e_0. \tag{165}$$

When the solution condition (163) is imposed, we get the equation

$$\partial_\theta b_{z_1} = \frac{1}{c}\partial_{x_1} e_0. \tag{166}$$

The equations for ϵ^2, (131), (136), (141) and (146), can be written as the system

$$\partial_\theta e_2 + c\partial_\theta b_{x_2} = -\partial_{\tau_1} e_1 - \partial_{\tau_2} e_0, \tag{167}$$
$$c\partial_\theta b_{z_2} = \partial_{x_1} e_1, \tag{168}$$
$$\partial_\theta b_{x_2} + \frac{1}{c}\partial_\theta e_2 = -\mu_0 c\partial_\theta p_2 - \partial_{\tau_1} b_{x_1} - \partial_{\tau_2} b_{x_0} + \partial_{x_1} b_{z_1}, \tag{169}$$
$$\partial_\theta b_{z_2} = -\partial_{x_1} b_{x_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}. \tag{170}$$

It's easy to see that the left side of equation (167) is equivalent to equation (169) multiplied by c. This means that in order for equation (167) and (169) to have a solution we have to impose the solvability condition

$$-\partial_{\tau_1} e_1 - \partial_{\tau_2} e_0 = -\mu_0 c^2 \partial_\theta p_2 - c\partial_{\tau_1} b_{x_1} - c\partial_{\tau_2} b_{x_0} + c\partial_{x_1} b_{z_1}. \tag{171}$$

In the same way is it for (168) and (170) to have a solution we have to impose the solvability condition

$$\partial_{x_1} e_1 = -c\partial_{x_1} b_{x_1} - c\partial_{\tau_1} b_{z_1} - c\partial_{\tau_2} b_{z_0}. \tag{172}$$

When the solution condition (171) is imposed, equation (167) becomes under-determined. This makes it possible to choose, without loss of generality

$$e_2 = 0. \tag{173}$$

Because we choose $e_1 = 0$ and $e_2 = 0$, equation (167) becomes

$$c\partial_\theta b_{x_2} = -\partial_{\tau_2} e_0. \tag{174}$$

And because of the solution condition (171) and that $e_1 = 0$, we get from equation (168) that

$$\partial_\theta b_{z_2} = 0. \tag{175}$$

The equations for ϵ^3, (132), (137), (142) and (147), can be written as the system

$$\partial_\theta e_3 + c\partial_\theta b_{x_3} = -\partial_{\tau_1} e_2 - \partial_{\tau_2} e_1 - \partial_{\tau_3} e_0, \tag{176}$$
$$c\partial_\theta b_{z_3} = \partial_{x_1} e_2, \tag{177}$$
$$\partial_\theta b_{x_3} + \frac{1}{c}\partial_\theta e_3 = -\mu_0 c\partial_\theta p_3 - \partial_{\tau_1} b_{x_2} - \partial_{\tau_2} b_{x_1} - \partial_{\tau_3} b_{x_0} + \partial_{x_1} b_{z_2}, \tag{178}$$
$$\partial_\theta b_{z_3} = -\partial_{x_1} b_{x_2} - \partial_{\tau_1} b_{z_2} - \partial_{\tau_2} b_{z_1} - \partial_{\tau_3} b_{z_0}. \tag{179}$$

We can see that the left side of equation (176) is the same as the left side of equation (178). This means that in order for (176) and (178) to have a solution, we have to impose the solvability condition

$$\begin{aligned} -\partial_{\tau_1} e_2 - \partial_{\tau_2} e_1 - \partial_{\tau_3} e_0 = -\mu_0 c^2 \partial_\theta p_3 - \partial_{\tau_1} b_{x_2} - \partial_{\tau_2} b_{x_1} \\ -\partial_{\tau_3} b_{x_0} + \partial_{x_1} b_{z_2}. \end{aligned} \tag{180}$$

In the same way as above it is necessary for (177) and (179) to have a solution, to impose the solvability condition

$$\partial_{x_1} e_2 = -c\partial_{x_1} b_{x_2} - c\partial_{\tau_1} b_{z_2} - c\partial_{\tau_2} b_{z_1} - \partial_{\tau_3} b_{z_0}. \tag{181}$$

When the solvability equation (180) is imposed (176) will become under-determined. This makes it possible without loss of generality to choose

$$e_3 = 0. \tag{182}$$

Because we have chosen $e_1 = 0$, $e_2 = 0$ and $e_3 = 0$, equation (176) will become

$$c\partial_\theta b_{x_3} = -\partial_{\tau_3} e_0. \tag{183}$$

Because of the solvability condition (181), and because we have chosen $e_2 = 0$, equation (177) becomes

$$\partial_\theta b_{z_3} = 0. \tag{184}$$

The equations for ϵ^4, (133), (138), (143) and (148), can be written as this system

$$\partial_\theta e_4 + c\partial_\theta b_{x_4} = -\partial_{\tau_1} e_3 - \partial_{\tau_2} e_2 - \partial_{\tau_3} e_1 - \partial_{\tau_4} e_0, \tag{185}$$

$$c\partial_\theta b_{z_4} = \partial_{x_1} e_3, \tag{186}$$

$$\begin{aligned} \partial_\theta b_{x_4} + \frac{1}{c}\partial_\theta e_4 = -\partial_{\tau_1} b_{x_3} - \partial_{\tau_2} b_{x_2} - \partial_{\tau_3} b_{x_1} \\ - \partial_{\tau_4} b_{x_0} + \partial_{x_1} b_{z_3} - \mu_0 c \partial_\theta p_4, \end{aligned} \tag{187}$$

$$\partial_\theta b_{z_4} = -\partial_{x_1} b_{x_3} - \partial_{\tau_1} b_{z_3} - \partial_{\tau_2} b_{z_2} - \partial_{\tau_3} b_{z_1} - \partial_{\tau_4} b_{z_0}. \tag{188}$$

We can easily see that equation (185) is the same as the left side of equation (187) under-determined by c. This means that in order for (185) and (187) to have a solution, we impose the solvability condition

$$\begin{aligned} -\partial_{\tau_1} e_3 - \partial_{\tau_2} e_2 - \partial_{\tau_3} e_1 - \partial_{\tau_4} e_0 &= -\partial_{\tau_1} b_{x_3} \\ -\partial_{\tau_2} b_{x_2} - \partial_{\tau_3} b_{x_1} - \partial_{\tau_4} b_{x_0} + \partial_{x_1} b_{z_3} - \mu_0 c \partial_\theta p_4. \end{aligned} \tag{189}$$

In the same way as above we will have to impose the solvability condition

$$\partial_{x_1} e_3 = -c\partial_{x_1} b_{x_3} - c\partial_{\tau_1} b_{z_3} - \partial_{\tau_2} b_{z_2} - \partial_{\tau_3} b_{z_1} - \partial_{\tau_4} b_{z_0}. \tag{190}$$

When the solvability condition (189) is imposed, equation (185) will become multivalued. This makes it possible to choose without loss of generality

$$e_4 = 0. \tag{191}$$

Because we have chosen $e_1 = 0$, $e_2 = 0$, $e_3 = 0$ and $e_4 = 0$, equation (185) becomes

$$c\partial_\theta b_{x_4} = -\partial_{\tau_4} e_0. \tag{192}$$

When the solvability condition (190) is imposed, and because $e_3 = 0$, we get from equation (186)

$$\partial_\theta b_{z_4} = 0. \tag{193}$$

As a summary we're now ending up with the equations (157), (155), (166), (165),(175), (174), (184), (183), (193), (192), and the solvability conditions (162), (163), (171), (172), (180), (181), (189) and (190). They are written as the system of equations

$$b_{z_0} = 0, \tag{194}$$

$$\partial_\theta b_{z_1} = \frac{1}{c}\partial_{x_1} e_0. \tag{195}$$

$$\partial_\theta b_{z_2} = 0, \tag{196}$$

$$\partial_\theta b_{z_3} = 0, \tag{197}$$

$$\partial_\theta b_{z_4} = 0, \tag{198}$$

$$b_{x_0} = -\frac{1}{c} e_0, \tag{199}$$

$$\partial_\theta b_{x_1} = -\frac{1}{c}\partial_{\tau_1} e_0, \tag{200}$$

$$c\partial_\theta b_{x_2} = -\partial_{\tau_2} e_0, \tag{201}$$

$$c\partial_\theta b_{x_3} = -\partial_{\tau_3} e_0, \tag{202}$$

$$c\partial_\theta b_{x_4} = -\partial_{\tau_4} e_0, \tag{203}$$

and the solvability conditions

$$
\begin{aligned}
&-\partial_{\tau_1} e_0 = -c\partial_{\tau_1} b_{x_0} - c\partial_{x_1} b_{z_0} - \mu_0 c^2 \partial_\theta p_1., && (204)\\
&\partial_{x_1} e_0 = -c\partial_{x_1} b_{x_0} - c\partial_{\tau_1} b_{z_0}, && (205)\\
&-\partial_{\tau_2} e_0 = -\mu_0 c^2 \partial_\theta p_2 - c\partial_{\tau_1} b_{x_1} - c\partial_{\tau_2} b_{x_0} \\
&+ c\partial_{x_1} b_{z_1}, && (206)\\
&0 = -\partial_{x_1} b_{x_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}, . && (207)\\
&-\partial_{\tau_3} e_0 = -\mu_0 c^2 \partial_\theta p_3 - \partial_{\tau_1} b_{x_2} - \partial_{\tau_2} b_{x_1} \\
&- \partial_{\tau_3} b_{x_0} + \partial_{x_1} b_{z_2}, && (208)\\
&0 = -c\partial_{x_1} b_{x_2} - c\partial_{\tau_1} b_{z_2} - c\partial_{\tau_2} b_{z_1} - \partial_{\tau_3} b_{z_0}, && (209)\\
&-\partial_{\tau_4} e_0 = -\partial_{\tau_1} b_{x_3} - \partial_{\tau_2} b_{x_2} \\
&- \partial_{\tau_3} b_{x_1} - \partial_{\tau_4} b_{x_0} + \partial_{x_1} b_{z_3} - \mu_0 c\partial_\theta p_4, && (210)
\end{aligned}
$$

Using equations (194) and (195) on (205) gives the equation

$$\partial_{x_1} e_0 = \partial_{x_1} e_0, \tag{211}$$

which is true. Using equation (200), (195) and (194) on (207) give us $0 = 0$. The same happens by doing the substitutions for (209), which means that the equations are automatically satisfied. Now, inserting equations (194) and (199) into equation (204) give the equation

$$2\partial_{\tau_1} e_0 = \mu_0 c^2 \partial_\theta p_1. \tag{212}$$

Finding the derivative on θ of equation (206), and inserting (200), (199) and (195) give the equation

$$2\partial_{\theta\tau_2} e_0 = \left(\mu_0 c^2 \partial_{\theta\theta} p_2 - \partial_{x_1 x_1} e_0\right). \tag{213}$$

Finding the derivative on θ of equation (208) and inserting (200), (195) and (199) give the equation

$$2\partial_{\theta\tau_3} e_0 = \mu_0 c^2 \partial_{\theta\theta} p_3. \tag{214}$$

And finding the derivative on θ of (210), and inserting equations (202), (201), (200), (199) and (197) give us the equation

$$2\partial_{\theta\tau_4} = \left(\mu_0 c^2 \partial_{\theta\theta} p_4 - \partial_{\tau_2\tau_2} e_0\right). \tag{215}$$

Consider the special case when

$$P = \epsilon^2 \left(\epsilon_0 \hat{\chi}(-ic\partial_\theta) e_0 + \epsilon_0 \eta e_0^3 \right),$$

which implies that

$$\begin{aligned} p_1 &= 0, \\ p_2 &= \left(\epsilon_0 \hat{\chi}(-ic\partial_\theta) e_0 + \epsilon_0 \eta e_0^3 \right), \\ p_3 &= 0, \\ p_4 &= 0. \end{aligned}$$

Introduce

$$E_0(\theta, x, \tau_1, \tau_2, ...)_{x_1 = \epsilon x, \tau_j = \epsilon^j \tau}. \tag{216}$$

Multiplying (212) by ϵ, (213) by ϵ^2, (214) by ϵ^3 and (215) by ϵ^4, adding and using the expansions

$$\partial_x = \epsilon \partial_{x_1}, \tag{217}$$
$$\partial_\tau = \epsilon \partial_{\tau_1} + \epsilon^2 \partial_{\tau_2} + \epsilon^3 \partial_{\tau_3} + \epsilon^4 \partial_{\tau_4}, \tag{218}$$

we get

$$2\partial_{\theta\tau} E_0 = \epsilon^2 \mu_0 c^2 \partial_{\theta\theta} \left(\epsilon_0 \hat{\chi}(-ic\partial_\theta) E_0 + \epsilon_0 \eta E_0^3 \right) - \partial_{xx} E_0 - \partial_{\tau\tau} E_0. \tag{219}$$

4.4 Vector Maxwell's equations order ϵ^4

Then let us finally start on the derivation of the perturbation equations. Starting all over again with Maxwell's equations

$$\begin{aligned} \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0, \\ \nabla \times \mathbf{B} &= \epsilon_0 \mu_0 \partial_t \mathbf{E} + \mu_0 \partial_t \mathbf{P}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{E} &= -\frac{1}{\epsilon_0} \nabla \cdot \mathbf{P}. \end{aligned}$$

This time are no assumptions made of the directions the polarization and the electric and magnetic fields have.

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}, \tag{220}$$
$$\mathbf{E} = E_x\mathbf{i} + E_y\mathbf{j} + E_z\mathbf{k}, \tag{221}$$
$$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}. \tag{222}$$

Inserting (220)-(222) into Maxwell's equations, starting with equation (1)

$$\begin{aligned}\nabla \times \mathbf{E} &= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} \\ &= (\partial_y E_z - \partial_z E_y)\,\mathbf{i} + (\partial_z E_x - \partial_x E_z)\,\mathbf{j} + (\partial_x E_y - \partial_y E_x)\,\mathbf{k},\end{aligned} \tag{223}$$

$$\begin{aligned}\nabla \times \mathbf{E} + \partial_t \mathbf{B} = (\partial_y E_z - \partial_z E_y)\,\mathbf{i} + (\partial_z E_x - \partial_x E_z)\,\mathbf{j} \\ + (\partial_x E_y - \partial_y E_x)\,\mathbf{k} + \partial_t B_x\mathbf{i} + \partial_t B_y\mathbf{j} + \partial_t B_z\mathbf{k} = 0.\end{aligned} \tag{224}$$

Then dividing this equation into vector components give the equations:

$$\partial_y E_z - \partial_z E_y + \partial_t B_x = 0, \tag{225}$$
$$\partial_z E_x - \partial_x E_z + \partial_t B_y = 0, \tag{226}$$
$$\partial_x E_y - \partial_y E_x + \partial_t B_z = 0. \tag{227}$$

Moving on to equation (2)

$$\begin{aligned}\nabla \times \mathbf{B} &= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= (\partial_y B_z + \partial_z B_y)\,\mathbf{i} + (\partial_z B_x - \partial_x B_z)\,\mathbf{j} + (\partial_x B_y - \partial_y B_x)\,\mathbf{k}.\end{aligned} \tag{228}$$

$$\begin{aligned}(\partial_y B_z + \partial_z B_y)\,\mathbf{i} + (\partial_z B_x - \partial_x B_z)\,\mathbf{j} + (\partial_x B_y - \partial_y B_x)\,\mathbf{k} \\ = \epsilon_0\mu_0\left(\partial_t E_x\mathbf{i} + \partial_t E_y\mathbf{j} + \partial_t E_z\mathbf{k}\right) + \mu_0\left(\partial_t P_x\mathbf{i} + \partial_t P_y\mathbf{j} + \partial_t P_z\mathbf{k}\right).\end{aligned} \tag{229}$$

Dividing equation (229) into vector components give the equations:

$$\partial_y B_z - \partial_z B_y = \epsilon_0\mu_0\partial_t E_x + \mu_0\partial_t P_x, \tag{230}$$
$$\partial_z B_x - \partial_x B_z = \epsilon_0\mu_0\partial_t E_y + \mu_0\partial_t P_y, \tag{231}$$
$$\partial_x B_y - \partial_y B_x = \epsilon_0\mu_0\partial_t E_z + \mu_0\partial_t P_z. \tag{232}$$

Equation (3) becomes

$$\nabla \cdot \mathbf{B} = 0,$$

$$\implies \partial_x B_x + \partial_y B_y + \partial_z B_z = 0, \tag{233}$$

and equation (4) becomes

$$\nabla \cdot \mathbf{E} = -\frac{1}{\epsilon_0} \nabla \cdot \mathbf{P},$$

$$\implies \partial_x E_x + \partial_y E_y + \partial_z E_z = -\frac{1}{\epsilon_0} \left(\partial_x P_x + \partial_y P_y + \partial_z P_z \right). \tag{234}$$

Ending up with a new set of equations, (225)-(227), (230)-(232), (233) and (234)

$$\begin{aligned}
&\partial_y E_z - \partial_z E_y + \partial_t B_x = 0, \\
&\partial_z E_x - \partial_x E_z + \partial_t B_y = 0, \\
&\partial_x E_y - \partial_y E_x + \partial_t B_z = 0, \\
&\partial_y B_z - \partial_z B_y = \epsilon_0 \mu_0 \partial_t E_x + \mu_0 \partial_t P_x, \\
&\partial_z B_x - \partial_x B_z = \epsilon_0 \mu_0 \partial_t E_y + \mu_0 \partial_t P_y, \\
&\partial_x B_y - \partial_y B_x = \epsilon_0 \mu_0 \partial_t E_z + \mu_0 \partial_t P_z, \\
&\partial_x B_x + \partial_y B_y + \partial_z B_z = 0, \\
&\partial_x E_x + \partial_y E_y + \partial_z E_z = -\frac{1}{\epsilon_0} \left(\partial_x P_x + \partial_y P_y + \partial_z P_z \right).
\end{aligned}$$

Introducing the change of variables (37)-(40)

$$\begin{aligned}
\theta &= z - ct, \\
\tau &= z, \\
\partial_z &= \frac{\partial \tau}{\partial z} \frac{\partial}{\partial \tau} + \frac{\partial \theta}{\partial z} \frac{\partial}{\partial \theta} = \partial_\tau + \partial_\theta, \\
\partial_t &= \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial \theta}{\partial t} \frac{\partial}{\partial \theta} = 0 + \partial_\theta \cdot = -c \partial_\theta.
\end{aligned}$$

Using these changes of variables on Maxwell's equations will give a new set of equations:

$$\partial_y E_z - \partial_\tau E_y - \partial_\theta E_y - c\partial_\theta B_x = 0, \tag{235}$$
$$\partial_\tau E_x + \partial_\theta E_x - \partial_x E_z - c\partial_\theta B_y = 0, \tag{236}$$
$$\partial_x E_y - \partial_y E_x - c\partial_\theta B_z = 0, \tag{237}$$
$$\partial_y B_z - \partial_\tau B_y - \partial_\theta B_y = -\frac{1}{c}\partial_\theta E_x - \mu_0 c\partial_\theta P_x, \tag{238}$$
$$\partial_\tau B_x + \partial_\theta B_x - \partial_x B_z = -\frac{1}{c}\partial_\theta E_y - \mu_0 c\partial_\theta P_y, \tag{239}$$
$$\partial_x B_y - \partial_y B_x = -\frac{1}{c}\partial_\theta E_z - \mu_0 c\partial_\theta P_z, \tag{240}$$
$$\partial_x B_x + \partial_y B_y + \partial_\tau B_z + \partial_\theta B_z = 0, \tag{241}$$
$$\partial_x E_x + \partial_y E_y + \partial_\tau E_z + \partial_\theta E_z = -\frac{1}{\epsilon_0}\left(\partial_x P_x + \partial_y P_y + \partial_z P_z\right). \tag{242}$$

4.4.1 The Multiple scale method

Introducing the equations

$$e_x = e_x(\theta, x_1, y_1, \tau_1, \tau_2), \tag{243}$$
$$e_y = e_y(\theta, x_1, y_1, \tau_1, \tau_2), \tag{244}$$
$$e_z = e_z(\theta, x_1, y_1, \tau_1, \tau_2), \tag{245}$$
$$b_x = b_x(\theta, x_1, y_1, \tau_1, \tau_2), \tag{246}$$
$$b_y = b_y(\theta, x_1, y_1, \tau_1, \tau_2), \tag{247}$$
$$b_z = b_z(\theta, x_1, y_1, \tau_1, \tau_2), \tag{248}$$
$$p_x = p_x(\theta, x_1, y_1, \tau_1, \tau_2), \tag{249}$$
$$p_y = p_y(\theta, x_1, y_1, \tau_1, \tau_2), \tag{250}$$
$$p_z = p_z(\theta, x_1, y_1, \tau_1, \tau_2). \tag{251}$$

where

$$x_1 = \epsilon x, \tag{252}$$
$$y_1 = \epsilon y, \tag{253}$$
$$\tau_j = \epsilon^j \tau. \tag{254}$$

Introducing the expansions

$$e_x = e_{x_0} + \epsilon e_{x_1} + \epsilon^2 e_{x_2} + \epsilon^3 e_{x_3} + \epsilon^4 e_{x_4} + ..., \tag{255}$$
$$e_y = e_{y_0} + \epsilon e_{y_1} + \epsilon^2 e_{y_2} + \epsilon^3 e_{y_3} + \epsilon^4 e_{y_4} + ..., \tag{256}$$
$$e_z = e_{z_0} + \epsilon e_{z_1} + \epsilon^2 e_{z_2} + \epsilon^3 e_{z_3} + \epsilon^4 e_{z_4} + ..., \tag{257}$$
$$b_x = b_{x_0} + \epsilon b_{x_1} + \epsilon^2 b_{x_2} + \epsilon^3 b_{x_3} + \epsilon^4 b_{x_4} + ..., \tag{258}$$
$$b_y = b_{y_0} + \epsilon b_{y_1} + \epsilon^2 b_{y_2} + \epsilon^3 b_{y_3} + \epsilon^4 b_{y_4} + ..., \tag{259}$$
$$b_z = b_{z_0} + \epsilon b_{z_1} + \epsilon^2 b_{z_2} + \epsilon^3 b_{z_3} + \epsilon^4 b_{z_4} + ..., \tag{260}$$
$$p_x = \epsilon p_{x_1} + \epsilon^2 p_{x_2} + \epsilon^3 p_{x_3} + \epsilon^4 p_{x_4} + ..., \tag{261}$$
$$p_y = \epsilon p_{y_1} + \epsilon^2 p_{y_2} + \epsilon^3 p_{y_3} + \epsilon^4 p_{y_4} + ..., \tag{262}$$
$$p_z = \epsilon p_{z_1} + \epsilon^2 p_{y_2} + \epsilon^3 p_{z_3} + \epsilon^4 p_{z_4} + ..., \tag{263}$$
$$\partial_\tau = \epsilon \partial_{\tau_1} + \epsilon^2 \partial_{\tau_2} + \epsilon^3 \partial_{\tau_3} + \epsilon^4 \ \partial_{\tau_4} + ..., \tag{264}$$
$$\partial_x = \epsilon \partial_{x_1}, \tag{265}$$
$$\partial_y = \epsilon \partial_{y_1}. \tag{266}$$

Inserting these expansions into equations (235)-(242), will make perturbation equations $f(\epsilon)$. Dividing these equations into parts that are multiplied with each order of ϵ, will make perturbation hierarchies. Starting with the perturbation hierarchy for equation (235):

$$\epsilon^0 : c\partial_\theta b_{x_0} + \partial_\theta e_{y_0} = 0, \tag{267}$$
$$\epsilon^1 : c\partial_\theta b_{x_1} + \partial_\theta e_{y_1} = \partial_{y_1} e_{z_0} - \partial_{\tau_1} e_{y_0}, \tag{268}$$
$$\epsilon^2 : c\partial_\theta b_{x_2} + \partial_\theta e_{y_2} = \partial_{y_1} e_{z_1} - \partial_{\tau_1} e_{y_1} - \partial_{\tau_2} e_{y_0}, \tag{269}$$
$$\epsilon^3 : c\partial_\theta b_{x_3} + \partial_\theta e_{y_3} = \partial_{y_1} e_{z_2} - \partial_{\tau_1} e_{y_2} - \partial_{\tau_2} e_{y_1} - \partial_{\tau_3} e_{y_0}, \tag{270}$$
$$\epsilon^4 : c\partial_\theta b_{x_4} + \partial_\theta e_{y_4} = \partial_{y_1} e_{z_3} - \partial_{\tau_1} e_{y_3} - \partial_{\tau_2} e_{y_2} - \partial_{\tau_3} e_{y_1} - \partial_{\tau_4} e_{y_0}. \tag{271}$$

The perturbation hierarchy for equation (236) is

$$\epsilon^0 : \partial_\theta e_{x_0} - c\partial_\theta b_{y_0} = 0, \tag{272}$$
$$\epsilon^1 : \partial_\theta e_{x_1} - c\partial_\theta b_{y_1} = \partial_{x_1} e_{z_0} - \partial_{\tau_1} e_{x_0}, \tag{273}$$
$$\epsilon^2 : \partial_\theta e_{x_2} - c\partial_\theta b_{y_2} = \partial_{x_1} e_{z_1} - \partial_{\tau_1} e_{x_1} - \partial_{\tau_2} e_{x_0}, \tag{274}$$
$$\epsilon^3 : \partial_\theta e_{x_3} - c\partial_\theta b_{y_3} = \partial_{x_1} e_{z_2} - \partial_{\tau_1} e_{x_2} - \partial_{\tau_2} e_{x_1} - \partial_{\tau_3} e_{x_0}, \tag{275}$$
$$\epsilon^4 : \partial_\theta e_{x_4} - c\partial_\theta b_{y_3} = \partial_{x_1} e_{z_3} - \partial_{\tau_1} e_{x_3} - \partial_{\tau_2} e_{x_2} - \partial_{\tau_3} e_{x_1} - \partial_{\tau_4} e_{x_0}. \tag{276}$$

The perturbation hierarchy for equation (237) is

$$\epsilon^0 : c\partial_\theta b_{z_0} = 0, \tag{277}$$
$$\epsilon^1 : c\partial_\theta b_{z_1} = \partial_{x_1} e_{y_0} - \partial_{y_1} e_{x_0}, \tag{278}$$
$$\epsilon^2 : c\partial_\theta b_{z_2} = \partial_{x_1} e_{y_1} - \partial_{y_1} e_{x_1}, \tag{279}$$
$$\epsilon^3 : c\partial_\theta b_{z_3} = \partial_{x_1} e_{y_2} - \partial_{y_1} e_{x_2}, \tag{280}$$
$$\epsilon^4 : c\partial_\theta b_{z_4} = \partial_{x_1} e_{y_3} - \partial_{y_1} e_{x_3}. \tag{281}$$

The perturbation hierarchy for equation (238) is

$$\epsilon^0 : \partial_\theta b_{y_0} - \frac{1}{c}\partial_\theta e_{x_0} = 0, \tag{282}$$
$$\epsilon^1 : \partial_\theta b_{y_1} - \frac{1}{c}\partial_\theta e_{x_1} = \partial_{y_1} b_{z_0} - \partial_{\tau_1} b_{y_0} + \mu_0 c\partial_\theta p_{x_1}, \tag{283}$$
$$\epsilon^2 : \partial_\theta b_{y_2} - \frac{1}{c}\partial_\theta e_{x_2} = \partial_{y_1} b_{z_1} - \partial_{\tau_1} b_{y_1} - \partial_{\tau_2} b_{y_0} + \mu_0 c\partial_\theta p_{x_2}, \tag{284}$$
$$\epsilon^3 : \partial_\theta b_{y_3} - \frac{1}{c}\partial_\theta e_{x_3} = \partial_{y_1} b_{z_2} - \partial_{\tau_1} b_{y_2} - \partial_{\tau_2} b_{y_1} - \partial_{\tau_3} b_{y_0} + \mu_0 c\partial_\theta p_{x_3}, \tag{285}$$
$$\epsilon^4 : \partial_\theta b_{y_4} - \frac{1}{c}\partial_\theta e_{x_4} = \partial_{y_1} b_{z_3} - \partial_{\tau_1} b_{y_3} - \partial_{\tau_2} b_{y_2} - \partial_{\tau_3} b_{y_1} - \partial_{\tau_4} b_{y_0} + \mu_0 c\partial_\theta p_{x_4}. \tag{286}$$

The perturbation hierarchy for equation (239) is

$$\epsilon^0 : \partial_\theta b_{x_0} + \frac{1}{c}\partial_\theta e_{y_0} = 0, \tag{287}$$
$$\epsilon^1 : \partial_\theta b_{x_1} + \frac{1}{c}\partial_\theta e_{y_1} = \partial_{x_1} b_{z_0} - \partial_{\tau_1} b_{x_0} - \mu_0 c\partial_\theta p_{y_1}, \tag{288}$$
$$\epsilon^2 : \partial_\theta b_{x_2} + \frac{1}{c}\partial_\theta e_{y_2} = \partial_{x_1} b_{z_1} - \partial_{\tau_1} b_{x_1} - \partial_{\tau_2} b_{x_0} - \mu_0 c\partial_\theta p_{y_2}, \tag{289}$$
$$\epsilon^3 : \partial_\theta b_{x_3} + \frac{1}{c}\partial_\theta e_{y_3} = \partial_{x_1} b_{z_2} - \partial_{\tau_1} b_{x_2} - \partial_{\tau_2} b_{x_1} - \partial_{\tau_3} b_{x_0} - \mu_0 c\partial_\theta p_{y_3}, \tag{290}$$
$$\epsilon^4 : \partial_\theta b_{x_4} + \frac{1}{c}\partial_\theta e_{y_4} = \partial_{x_1} b_{z_3} - \partial_{\tau_1} b_{x_3} - \partial_{\tau_2} b_{x_2} - \partial_{\tau_3} b_{x_1} - \partial_{\tau_4} b_{x_0} - \mu_0 c\partial_\theta p_{y_4}. \tag{291}$$

The perturbation hierarchy for equation (240) is

$$\epsilon^0 : \frac{1}{c}\partial_\theta e_{z_0} = 0, \tag{292}$$

$$\epsilon^1 : \frac{1}{c}\partial_\theta e_{z_1} = \partial_{y_1} b_{x_0} - \partial_{x_1} b_{y_0} - \mu_0 c \partial_\theta p_{z_1}, \tag{293}$$

$$\epsilon^2 : \frac{1}{c}\partial_\theta e_{z_2} = \partial_{y_1} b_{x_1} - \partial_{x_1} b_{y_1} - \mu_0 c \partial_\theta p_{z_2}, \tag{294}$$

$$\epsilon^3 : \frac{1}{c}\partial_\theta e_{z_3} = \partial_{y_1} b_{x_2} - \partial_{x_1} b_{y_2} - \mu_0 c \partial_\theta p_{z_3}, \tag{295}$$

$$\epsilon^4 : \frac{1}{c}\partial_\theta e_{z_4} = \partial_{y_1} b_{x_3} - \partial_{x_1} b_{y_3} - \mu_0 c \partial_\theta p_{z_4}. \tag{296}$$

The perturbation hierarchy for equation (241) is

$$\epsilon^0 : \partial_\theta b_{z_0} = 0, \tag{297}$$

$$\epsilon^1 : \partial_\theta b_{z_1} = -\partial_{x_1} b_{x_0} - \partial_{y_1} b_{y_0} - \partial_{\tau_1} b_{z_0}, \tag{298}$$

$$\epsilon^2 : \partial_\theta b_{z_2} = -\partial_{x_1} b_{x_1} - \partial_{y_1} b_{y_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}, \tag{299}$$

$$\epsilon^3 : \partial_\theta b_{z_3} = -\partial_{x_1} b_{x_2} - \partial_{y_1} b_{y_2} - \partial_{\tau_1} b_{z_2} - \partial_{\tau_2} b_{z_1} - \partial_{\tau_3} b_{z_0}, \tag{300}$$

$$\epsilon^4 : \partial_\theta b_{z_4} = -\partial_{x_1} b_{x_3} - \partial_{y_1} b_{y_3} - \partial_{\tau_1} b_{z_3} - \partial_{\tau_2} b_{z_2} - \partial_{\tau_3} b_{z_1} - \partial_{\tau_4} b_{z_0}. \tag{301}$$

And the perturbation hierarchy for equation (242) is

$$\epsilon^0 : \partial_\theta e_{z_0} = 0, \tag{302}$$

$$\epsilon^1 : \partial_\theta e_{z_1} = -\partial_{x_1} e_{x_0} - \partial_{y_1} e_{y_0} - \partial_{\tau_1} e_{z_0} - \frac{1}{\epsilon_0}\partial_\theta p_{z_1}, \tag{303}$$

$$\begin{aligned}\epsilon^2 : \partial_\theta e_{z_2} = &-\partial_{x_1} e_{x_1} - \partial_{y_1} e_{y_1} - \partial_{\tau_1} e_{z_1} - \partial_{\tau_2} e_{z_0} - \frac{1}{\epsilon_0}\partial_{x_1} p_{x_1} \\ &- \frac{1}{\epsilon_0}\partial_{y_1} p_{y_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_1} - \frac{1}{\epsilon_0}\partial_\theta p_{z_2},\end{aligned} \tag{304}$$

$$\begin{aligned}\epsilon^3 : \partial_\theta e_{z_3} = &-\partial_{x_1} e_{x_2} - \partial_{y_1} e_{y_2} - \partial_{\tau_1} e_{z_2} - \partial_{\tau_2} e_{z_1} - \partial_{\tau_3} e_{z_0} \\ &- \frac{1}{\epsilon_0}\partial_{x_1} p x_2 - \frac{1}{\epsilon_0}\partial_{y_1} p_{y_2} - \frac{1}{\epsilon_0}\partial_{\tau_2} p_{z_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_2} - \frac{1}{\epsilon_0}\partial_\theta p_{z_3},\end{aligned} \tag{305}$$

$$\begin{aligned}\epsilon^4 : \partial_\theta e_{z_4} = &-\partial_{x_1} e_{x_3} - \partial_{y_1} e_{y_3} - \partial_{\tau_1} e_{z_3} - \partial_{\tau_2} e_{z_2} - \partial_{\tau_3} e_{z_1} \\ &- \partial_{\tau_4} e_{z_0} - \frac{1}{\epsilon_0}\partial_{x_1} p_{x_3} - \frac{1}{\epsilon_0}\partial_{y_1} p_{y_3} - \frac{1}{\epsilon_0}\partial_{\tau_3} p_{z_1} - \frac{1}{\epsilon_0}\partial_{\tau_2} p_{z_2} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_3} \\ &- \frac{1}{\epsilon_0}\partial_\theta p_{z_4}.\end{aligned} \tag{306}$$

The way to solve these equations is to divide them into groups from what order of ϵ they belong to, and then solve them like that. Starting with ϵ^0, (267), (272), (277), (282), (287), (292), (297) and (302), and writing them as the system

$$c\partial_\theta b_{x_0} + \partial_\theta e_{y_0} = 0, \tag{307}$$
$$\partial_\theta e_{x_0} - c\partial_\theta b_{y_0} = 0, \tag{308}$$
$$c\partial_\theta b_{z_0} = 0, \tag{309}$$
$$\partial_\theta b_{y_0} - \frac{1}{c}\partial_\theta e_{x_0} = 0, \tag{310}$$
$$\partial_\theta b_{x_0} + \frac{1}{c}\partial_\theta e_{y_0} = 0, \tag{311}$$
$$\frac{1}{c}\partial_\theta e_{z_0} = 0, \tag{312}$$
$$\partial_\theta b_{z_0} = 0, \tag{313}$$
$$\partial_\theta e_{z_0} = 0. \tag{314}$$

Two and two of these equations are equal, so it's actually four equations, (307), (308), (313) and (314):

$$\begin{aligned} c\partial_\theta b_{x_0} + \partial_\theta e_{y_0} &= 0, \\ \partial_\theta e_{x_0} - c\partial_\theta b_{y_0} &= 0, \\ \partial_\theta b_{z_0} &= 0, \\ \partial_\theta e_{z_0} &= 0. \end{aligned}$$

Equation (304) has the general solution

$$cb_{x_0} + e_{y_0} = a(x_1, y_1, \tau_1, \tau_2, ...). \tag{315}$$

Since a doesn't depend on θ can it be disregarded, and we choose $a = 0$

$$cb_{x_0} + e_{y_0} = 0,$$

$$\implies b_{x_0} = -\frac{1}{c}e_{y_0}. \tag{316}$$

Equation (308) has the general solution

$$e_{x_0} - cb_{y_0} = d(x_1, y_1, \tau_1, \tau_2, ...). \tag{317}$$

Because d doesn't depend on θ, can we as before choose $d = 0$, and equation (308) has the solution

$$b_{y_0} = \frac{1}{c}e_{x_0}. \tag{318}$$

Equation (309) has the general solution

$$b_{z_0} = f(x_1, y_1, \tau_1, \tau_2, ...). \tag{319}$$

Because f does not depend on θ, can we choose $f = 0$, and

$$b_{z_0} = 0. \tag{320}$$

The general solution to equation (310) is

$$e_{z_0} = g(x_1, y_1, \tau_1, \tau_2, ...). \tag{321}$$

Since g doesn't depend on θ, will the solution to (310) be

$$e_{z_0} = 0. \tag{322}$$

And then moving on to the equations for ϵ^1, (268), (273), (278), (283), (288), (293), (298) and (303), which can be written as the system

$$c\partial_\theta b_{x_1} + \partial_\theta e_{y_1} = \partial_{y_1} e_{z_0} - \partial_{\tau_1} e_{y_0}, \tag{323}$$
$$\partial_\theta e_{x_1} - c\partial_\theta b_{y_1} = \partial_{x_1} e_{z_0} - \partial_{\tau_1} e_{x_0}, \tag{324}$$
$$c\partial_\theta b_{z_1} = \partial_{x_1} e_{y_0} - \partial_{y_1} e_{x_0}, \tag{325}$$
$$\partial_\theta b_{y_1} - \frac{1}{c}\partial_\theta e_{x_1} = \partial_{y_1} b_{z_0} - \partial_{\tau_1} b_{y_0} + \mu_0 c\partial_\theta p_{x_1}, \tag{326}$$
$$\partial_\theta b_{x_1} + \frac{1}{c}\partial_\theta e_{y_1} = \partial_{x_1} b_{z_0} - \partial_{\tau_1} b_{x_0} - \mu_0 c\partial_\theta p_{y_1}, \tag{327}$$
$$\frac{1}{c}\partial_\theta e_{z_1} = \partial_{y_1} b_{x_0} - \partial_{x_1} b_{y_0} - \mu_0 c\partial_\theta p_{z_1}, \tag{328}$$
$$\partial_\theta b_{z_1} = -\partial_{x_1} b_{x_0} - \partial_{y_1} b_{y_0} - \partial_{\tau_1} b_{z_0}, \tag{329}$$
$$\partial_\theta e_{z_1} = -\partial_{x_1} e_{x_0} - \partial_{y_1} e_{y_0} - \partial_{\tau_1} e_{z_0} - \frac{1}{\epsilon_0}\partial_\theta p_{z_1}. \tag{330}$$

Let us start with looking at equations (323) and (327). It's easy to see that the left side of equation (323) is equal to the left side of (327) multiplied by c. In order for (323) and (327) to have a solution, we have to impose the solvability condition

$$\partial_{y_1} e_{z_0} - \partial_{\tau_1} e_{y_0} = c(\partial_{x_1} b_{z_0} - \partial_{\tau_1} b_{x_0} - \mu_0 c\partial_\theta p_{y_1}). \tag{331}$$

In the same way as above equation (324) and (326) will only have a solution when we impose the solvability condition

$$\partial_{x_1} e_{z_0} - \partial_{\tau_1} e_{x_0} = c(\partial_{y_1} b_{z_0} - \partial_{\tau_1} b_{y_0} + \mu_0 c \partial_\theta p_{x_1}). \tag{332}$$

Moving on to equation (325) and (329). They will only have a solution if we impose the solvability condition

$$\partial_{x_1} e_{y_0} - \partial_{y_1} e_{x_0} = c(-\partial_{x_1} b_{x_0} - \partial_{y_1} b_{y_0} - \partial_{\tau_1} b_{z_0}). \tag{333}$$

And equations (328) and equation (330) only have a solution if we impose the solvability condition

$$-\partial_{x_1} e_{x_0} - \partial_{y_1} e_{y_0} - \partial_{\tau_1} e_{z_0} - \frac{1}{\epsilon_0} \partial_\theta p_{z_1} = c(\partial_{y_1} b_{x_0} - \partial_{x_1} b_{y_0} - \mu_0 c \partial_\theta p_{z_1}). \tag{334}$$

Because of the solvability condition (331) equation (323) will become underdetermined. This makes it possible to choose without loss of generality

$$e_{y_1} = 0. \tag{335}$$

When $e_{y_1} = 0$, equation (323) becomes

$$c\partial_\theta b_{x_1} = \partial_{y_1} e_{z_0} - \partial_{\tau_1} e_{y_0}. \tag{336}$$

The solvability condition (332) makes equation (324) become multivalued. This means that we can choose without loss of generality

$$e_{x_1} = 0. \tag{337}$$

When $e_{x_1} = 0$ equation (324) becomes

$$-c\partial_\theta b_{y_1} = \partial_{x_1} e_{z_0} - \partial_{\tau_1} e_{x_0}. \tag{338}$$

Imposing the solvability conditions (333) and (333) give us

$$c\partial_\theta b_{z_1} = \partial_{x_1} e_{y_0} - \partial_{y_1} e_{x_0}, \tag{339}$$

and

$$\partial_\theta e_{z_1} = -\partial_{x_1} e_{x_0} - \partial_{y_1} e_{y_0} - \partial_{\tau_1} e_{z_0} - \frac{1}{\epsilon_0} \partial_\theta p_{z_1}. \tag{340}$$

Moving on to solve the equations for ϵ^2, (269), (274), (279), (284), (289), (294), (299) and (304), which can be written as the system

$$
\begin{aligned}
c\partial_\theta b_{x_2} + \partial_\theta e_{y_2} &= \partial_{y_1} e_{z_1} - \partial_{\tau_1} e_{y_1} - \partial_{\tau_2} e_{y_0} && (341)\\
\partial_\theta e_{x_2} - c\partial_\theta b_{y_2} &= \partial_{x_1} e_{z_1} - \partial_{\tau_1} e_{x_1} - \partial_{\tau_2} e_{x_0}, && (342)\\
c\partial_\theta b_{z_2} &= \partial_{x_1} e_{y_1} - \partial_{y_1} e_{x_1}, && (343)\\
\partial_\theta b_{y_2} - \frac{1}{c}\partial_\theta e_{x_2} &= \partial_{y_1} b_{z_1} - \partial_{\tau_1} b_{y_1} - \partial_{\tau_2} b_{y_0} + \mu_0 c\partial_\theta p_{x_2}, && (344)\\
\partial_\theta b_{x_2} + \frac{1}{c}\partial_\theta e_{y_2} &= \partial_{x_1} b_{z_1} - \partial_{\tau_1} b_{x_1} - \partial_{\tau_2} b_{x_0} + \mu_0 c\partial_\theta p_{y_2}, && (345)\\
\frac{1}{c}\partial_\theta e_{z_2} &= \partial_{y_1} b_{x_1} - \partial_{x_1} b_{y_1} - \mu_0 c\partial_\theta p_{z_2}, && (346)\\
\partial_\theta b_{z_2} &= -\partial_{x_1} b_{x_1} - \partial_{y_1} b_{y_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}, && (347)\\
\partial_\theta e_{z_2} &= -\partial_{x_1} e_{x_1} - \partial_{y_1} e_{y_1} - \partial_{\tau_2} e_{z_0} - \partial_{\tau_1} e_{z_1} - \frac{1}{\epsilon_0}\partial_{x_1} p_{x_1} \\
&\quad - \frac{1}{\epsilon_0}\partial_{y_1} p_{y_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_1} - \frac{1}{\epsilon_0}\partial_\theta p_{z_2}. && (348)
\end{aligned}
$$

It's easy to see that equation (341) is (345) multiplied by c. So in order for equation (341) and (345) to have a solution we impose the solvability condition

$$\partial_{y_1} e_{z_1} - \partial_{\tau_1} e_{y_1} - \partial_{\tau_2} e_{y_0} = c(\partial_{x_1} b_{z_1} - \partial_{\tau_1} b_{x_1} - \partial_{\tau_2} b_{x_0} + \mu_0 c\partial_\theta p_{y_2}). \quad (349)$$

In the same way as above will equation (342) and (344) only have a solution if we impose the solvability condition

$$\partial_{x_1} e_{z_1} - \partial_{\tau_1} e_{x_1} - \partial_{\tau_2} e_{x_0} = c(\partial_{y_1} b_{z_1} - \partial_{\tau_1} b_{y_1} - \partial_{\tau_2} b_{y_0} + \mu_0 c\partial_\theta p_{x_2}). \quad (350)$$

In the same way will also equation (343) and (347) only have a solution if we impose the solvability condition

$$\partial_{x_1} e_{y_1} - \partial_{y_1} e_{x_1} = c(-\partial_{x_1} b_{x_1} - \partial_{y_1} b_{y_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}), \quad (351)$$

And equation (346) and (348) needs the solvability condition

$$
\begin{aligned}
&\partial_{x_1} e_{x_1} - \partial_{y_1} e_{y_1} - \partial_{\tau_2} e_{z_0} - \partial_{\tau_1} e_{z_1} \\
&- \frac{1}{\epsilon_0}\partial_{x_1} p_{x_1} - \frac{1}{\epsilon_0}\partial_{y_1} p_{y_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_1} - \frac{1}{\epsilon_0}\partial_\theta p_{z_2} \qquad (352)\\
&= c(\partial_{y_1} b_{x_1} - \partial_{x_1} b_{y_1} - \mu_0 c\partial_\theta p_{z_2}).
\end{aligned}
$$

When the solvability condition (349) is imposed, equation (341) becomes under-determied. This means that it is possible without loss of generality to choose

$$e_{y_2} = 0. \tag{353}$$

When $e_{y_2} = 0$, equation (341) becomes

$$c\partial_\theta b_{x_2} = \partial_{y_1} e_{z_1} - \partial_{\tau_1} e_{y_1} - \partial_{\tau_2} e_{y_0}. \tag{354}$$

When the solvability condition (350) is imposed, equation (342) becomes under-determined. This means that it is possible to choose without loss of generality

$$e_{x_2} = 0. \tag{355}$$

When $e_{x_2} = 0$ equation (342) becomes

$$-c\partial_\theta b_{y_2} = \partial_{x_1} e_{z_1} - \partial_{\tau_1} e_{x_1} - \partial_{\tau_2} e_{x_0}. \tag{356}$$

When the solvability conditions (351) and (352) are imposed we will get the equations

$$c\partial_\theta b_{z_2} = \partial_{x_1} e_{y_1} - \partial_{y_1} e_{x_1}, \tag{357}$$

and

$$\begin{aligned} \partial_\theta e_{z_2} = &-\partial_{x_1} e_{x_1} - \partial_{y_1} e_{y_1} - \partial_{\tau_2} e_{z_0} - \partial_{\tau_1} e_{z_1} - \frac{1}{\epsilon_0}\partial_{x_1} p_{x_1} \\ &- \frac{1}{\epsilon_0}\partial_{y_1} p_{y_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_1} - \frac{1}{\epsilon_0}\partial_\theta p_{z_2}. \end{aligned} \tag{358}$$

The equations for when ϵ^3, (270), (275), (280), (285), (290), (295), (300) and (305) can be written as the system

$$c\partial_\theta b_{x_3} + \partial_\theta e_{y_3} = \partial_{y_1} e_{z_2} - \partial_{\tau_1} e_{y_2} - \partial_{\tau_2} e_{y_1} - \partial_{\tau_3} e_{y_0}, \tag{359}$$

$$\partial_\theta e_{x_3} - c\partial_\theta b_{y_3} = \partial_{x_1} e_{z_2} - \partial_{\tau_1} e_{x_2} - \partial_{\tau_2} e_{x_1} - \partial_{\tau_3} e_{x_0}, \tag{360}$$

$$c\partial_\theta b_{z_3} = \partial_{x_1} e_{y_2} - \partial_{y_1} e_{x_2}, \tag{361}$$

$$\partial_\theta b_{y_3} - \frac{1}{c}\partial_\theta e_{x_3} = \partial_{y_1} b_{z_2} - \partial_{\tau_1} b_{y_2} - \partial_{\tau_2} b_{y_1} - \partial_{\tau_3} b_{y_0} + \mu_0 c\partial_\theta p_{x_3}, \tag{362}$$

$$\partial_\theta b_{x_3} + \frac{1}{c}\partial_\theta e_{y_3} = \partial_{x_1} b_{z_2} - \partial_{\tau_1} b_{x_2} - \partial_{\tau_2} b_{x_1} - \partial_{\tau_3} b_{x_0} - \mu_0 c\partial_\theta p_{y_3}, \tag{363}$$

$$\frac{1}{c}\partial_\theta e_{z_3} = \partial_{y_1} b_{x_2} - \partial_{x_1} b_{y_2} - \mu_0 c\partial_\theta p_{z_3}, \tag{364}$$

$$\partial_\theta b_{z_3} = -\partial_{x_1} b_{x_2} - \partial_{y_1} b_{y_2} - \partial_{\tau_1} b_{z_2} - \partial_{\tau_2} b_{z_1} - \partial_{\tau_3} b_{z_0}, \tag{365}$$

$$\begin{aligned}\partial_\theta e_{z_3} = &-\partial_{x_1} e_{x_2} - \partial_{y_1} e_{y_2} - \partial_{\tau_3} e_{z_0} - \partial_{\tau_2} e_{z_1} - \partial_{\tau_1} e_{z_2} \\ &- \frac{1}{\epsilon_0}\partial_{x_1} p_{x_2} - \frac{1}{\epsilon_0}\partial_{y_1} p_{y_2} - \frac{1}{\epsilon_0}\partial_{\tau_2} p_{z_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_2} \\ &- \frac{1}{\epsilon_0}\partial_\theta p_{z_3}.\end{aligned} \tag{366}$$

In order for (359) and (363) to have a solution we will impose the solvability condition

$$\begin{aligned}&\partial_{y_1} e_{z_2} - \partial_{\tau_1} e_{y_2} - \partial_{\tau_2} e_{y_1} - \partial_{\tau_3} e_{y_0} \\ &= c(\partial_{x_1} b_{z_2} - \partial_{\tau_1} b_{x_2} - \partial_{\tau_2} b_{x_1} - \partial_{\tau_3} b_{x_0} - \mu_0 c\partial_\theta p_{y_3}).\end{aligned} \tag{367}$$

In order for (360) and (362) to have a solution we impose the solvability condition

$$\begin{aligned}&\partial_{x_1} e_{z_2} - \partial_{\tau_1} e_{x_2} - \partial_{\tau_2} e_{x_1} - \partial_{\tau_3} e_{x_0} \\ &= \partial_{y_1} b_{z_2} - \partial_{\tau_1} b_{y_2} - \partial_{\tau_2} b_{y_1} - \partial_{\tau_3} b_{y_0} + \mu_0 c\partial_\theta p_{x_3}.\end{aligned} \tag{368}$$

In order for (361) and (365) to have a solution we impose the solvability condition

$$\partial_{x_1} e_{y_2} - \partial_{y_1} e_{x_2} = c(-\partial_{x_1} b_{x_2} - \partial_{y_1} b_{y_2} - \partial_{\tau_1} b_{z_2} - \partial_{\tau_2} b_{z_1} - \partial_{\tau_3} b_{z_0}). \tag{369}$$

And in order for (364) and (366) to have a solution we impose the solvability condition

$$\begin{aligned}&- \partial_{x_1} e_{x_2} - \partial_{y_1} e_{y_2} - \partial_{\tau_3} e_{z_0} - \partial_{\tau_2} e_{z_1} - \partial_{\tau_1} e_{z_2} \\ &- \frac{1}{\epsilon_0}\partial_{x_1} p_{x_2} - \frac{1}{\epsilon_0}\partial_{y_1} p_{y_2} - \frac{1}{\epsilon_0}\partial_{\tau_2} p_{z_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_2} \\ &- \frac{1}{\epsilon_0}\partial_\theta p_{z_3} = c(\partial_{y_1} b_{x_2} - \partial_{x_1} b_{y_2} - \mu_0 c\partial_\theta p_{z_3}).\end{aligned} \tag{370}$$

When the solvability condition (367) is imposed, equation (359) becomes under-determined, and it's possible without loss of generality to choose

$$e_{y_3} = 0. \tag{371}$$

When $e_{y_3} = 0$ equation (359) becomes

$$c\partial_\theta b_{x_3} = \partial_{y_1} e_{z_2} - \partial_{\tau_1} e_{y_2} - \partial_{\tau_2} e_{y_1} - \partial_{\tau_3} e_{y_0}. \tag{372}$$

When the solvability condition (368) is imposed equation (360) becomes under-determined, and it is possible to choose without loss of generality

$$e_{x_3} = 0. \tag{373}$$

When $e_{x_3} = 0$, equation (360) becomes

$$-c\partial_\theta b_{y_3} = \partial_{x_1} e_{z_2} - \partial_{\tau_1} e_{x_2} - \partial_{\tau_2} e_{x_1} - \partial_{\tau_3} e_{x_0}. \tag{374}$$

And when the solvability conditions (369) and (370) are imposed, we get the equations

$$c\partial_\theta b_{z_3} = \partial_{x_1} e_{y_2} - \partial_{y_1} e_{x_2}, \tag{375}$$

and

$$\begin{aligned} \partial_\theta e_{z_3} = &-\partial_{x_1} e_{x_2} - \partial_{y_1} e_{y_2} - \partial_{\tau_3} e_{z_0} - \partial_{\tau_2} e_{z_1} - \partial_{\tau_1} e_{z_2} \\ &- \frac{1}{\epsilon_0}\partial_{x_1} p_{x_2} - \frac{1}{\epsilon_0}\partial_{y_1} p_{y_2} - \frac{1}{\epsilon_0}\partial_{\tau_2} p_{z_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_2} \\ &- \frac{1}{\epsilon_0}\partial_\theta p_{z_3}. \end{aligned} \tag{376}$$

And then the last set of equations for ϵ^4, (271), (276), (281), (286), (291), (296), (301) and (306), which can be written as this following system:

$$c\partial_\theta b_{x_4} + \partial_\theta e_{y_4} = \partial_{y_1} e_{z_3} - \partial_{\tau_1} e_{y_3} - \partial_{\tau_2} e_{y_2} - \partial_{\tau_3} e_{y_1} - \partial_{\tau_4} e_{y_0}, \tag{377}$$

$$\partial_\theta e_{x_4} - c\partial_\theta b_{y_4} = \partial_{x_1} e_{z_3} - \partial_{\tau_1} e_{x_3} - \partial_{\tau_2} e_{x_2} - \partial_{\tau_3} e_{x_1} - \partial_{\tau_4} e_{x_0}, \tag{378}$$

$$c\partial_\theta b_{z_4} = \partial_{x_1} e_{y_3} - \partial_{y_1} e_{x_3}, \tag{379}$$

$$\begin{aligned}\partial_\theta b_{y_4} - \frac{1}{c}\partial_\theta e_{x_4} &= \partial_{y_1} b_{z_3} - \partial_{\tau_1} b_{y_3} - \partial_{\tau_2} b_{y_2} - \partial_{\tau_3} b_{y_1} \\ &- \partial_{\tau_4} b_{y_0} + \mu_0 c\partial_\theta p_{x_4},\end{aligned} \tag{380}$$

$$\begin{aligned}\partial_\theta b_{x_4} + \frac{1}{c}\partial_\theta e_{y_4} &= \partial_{x_1} b_{z_3} - \partial_{\tau_1} b_{x_3} - \partial_{\tau_2} b_{x_2} - \partial_{\tau_3} b_{x_1} \\ &- \partial_{\tau_4} b_{x_0} - \mu_0 c\partial_\theta p_{y_4},\end{aligned} \tag{381}$$

$$\frac{1}{c}\partial_\theta e_{z_4} = \partial_{y_1} b_{x_3} - \partial_{x_1} b_{y_3} - \mu_0 c\partial_\theta p_{z_4}, \tag{382}$$

$$\begin{aligned}\partial_\theta b_{z_4} &= -\partial_{x_1} b_{x_3} - \partial_{y_1} b_{y_3} - \partial_{\tau_1} b_{z_3} - \partial_{\tau_2} b_{z_2} \\ &- \partial_{\tau_3} b_{z_1} - \partial_{\tau_4} b_{z_0},\end{aligned} \tag{383}$$

$$\begin{aligned}\partial_\theta e_{z_4} &= -\partial_{x_1} e_{x_3} - \partial_{y_1} e_{y_3} - \partial_{\tau_4} e_{z_0} - \partial_{\tau_3} e_{z_1} \\ &- \partial_{\tau_2} e_{z_2} - \partial_{\tau_1} e_{z_3} - \frac{1}{\epsilon_0}\partial_{x_1} p_{x_3} \\ &- \frac{1}{\epsilon_0}\partial_{y_1} p_{y_3} - \frac{1}{\epsilon_0}\partial_{\tau_3} p_{z_1} \\ &- \frac{1}{\epsilon_0}\partial_{\tau_2} p_{z_2} - \frac{1}{\epsilon_0}\partial\tau_1 p_{z_3} - \frac{1}{\epsilon_0}\partial_\theta p_{z_4}.\end{aligned} \tag{384}$$

In order for (377) and (381) to have a solution we impose the solvability condition

$$\begin{aligned}&\partial_{y_1} e_{z_3} - \partial_{\tau_1} e_{y_3} - \partial_{\tau_2} e_{y_2} - \partial_{\tau_3} e_{y_1} - \partial_{\tau_4} e_{y_0} \\ &= c(\partial_{x_1} b_{z_3} - \partial_{\tau_1} b_{x_3} - \partial_{\tau_2} b_{x_2} - \partial_{\tau_3} b_{x_1} - \partial_{\tau_4} b_{x_0} - \mu_0 c\partial_\theta p_{y_4}).\end{aligned} \tag{385}$$

In order for (378) and (380) to have a solution we impose the solvability condition

$$\begin{aligned}&\partial_{x_1} e_{z_3} - \partial_{\tau_1} e_{x_3} - \partial_{\tau_2} e_{x_2} - \partial_{\tau_3} e_{x_1} - \partial_{\tau_4} e_{x_0} \\ &= c(\partial_{y_1} b_{z_3} - \partial_{\tau_1} b_{y_3} - \partial_{\tau_2} b_{y_2} - \partial_{\tau_3} b_{y_1} \\ &- \partial_{\tau_4} b_{y_0} + \mu_0 c\partial_\theta p_{x_4}).\end{aligned} \tag{386}$$

In order for (379) and (383) to have a solution we impose the solvability condition

$$\begin{aligned}&\partial_{x_1} e_{y_3} - \partial_{y_1} e_{x_3} \\ &= c(-\partial_{x_1} b_{x_3} - \partial_{y_1} b_{y_3} - \partial_{\tau_1} b_{z_3} - \partial_{\tau_2} b_{z_2} - \partial_{\tau_3} b_{z_1} - \partial_{\tau_4} b_{z_0}).\end{aligned} \tag{387}$$

And in order for (382) and (384) to have a solution we impose the solvability condition

$$\begin{aligned}
&-\partial_{x_1} e_{x_3} - \partial_{y_1} e_{y_3} - \partial_{\tau_4} e_{z_0} - \partial_{\tau_3} e_{z_1} \\
&- \partial_{\tau_2} e_{z_2} - \partial_{\tau_1} e_{z_3} - \frac{1}{\epsilon_0}\partial_{x_1} p_{x_3} \\
&- \frac{1}{\epsilon_0}\partial_{y_1} p_{y_3} - \frac{1}{\epsilon_0}\partial_{\tau_3} p_{z_1} \\
&- \frac{1}{\epsilon_0}\partial_{\tau_2} p_{z_2} - \frac{1}{\epsilon_0}\partial \tau_1 p_{z_3} - \frac{1}{\epsilon_0}\partial_\theta p_{z_4} \\
&= c(\partial_{y_1} b_{x_3} - \partial_{x_1} b_{y_3} - \mu_0 c \partial_\theta p_{z_4}).
\end{aligned} \tag{388}$$

When the solvability condition (385) is imposed, equation (377) becomes under-determined. This makes it possible to without loss of generality choose

$$e_{y_4} = 0. \tag{389}$$

When $e_{y_4} = 0$ equation (377) becomes

$$c\partial_\theta b_{x_4} = \partial_{y_1} e_{z_3} - \partial_{\tau_1} e_{y_3} - \partial_{\tau_2} e_{y_2} - \partial_{\tau_3} e_{y_1} - \partial_{\tau_4} e_{y_0}. \tag{390}$$

When the solvability condition (386) is imposed, equation (378) becomes under-determined and we can choose without loss of generality

$$e_{x_4} = 0. \tag{391}$$

When $e_{x_4} = 0$ equation (378) becomes

$$-c\partial_\theta b_{y_4} = \partial_{x_1} e_{z_3} - \partial_{\tau_1} e_{x_3} - \partial_{\tau_2} e_{x_2} - \partial_{\tau_3} e_{x_1} - \partial_{\tau_4} e_{x_0}. \tag{392}$$

And when the solvability conditions (387) and (388) are imposed, we get the equations

$$c\partial_\theta b_{z_4} = \partial_{x_1} e_{y_3} - \partial_{y_1} e_{x_3}, \tag{393}$$

and

$$\begin{aligned}
&\partial_\theta e_{z_4} = -\partial_{x_1} e_{x_3} - \partial_{y_1} e_{y_3} - \partial_{\tau_4} e_{z_0} - \partial_{\tau_3} e_{z_1} \\
&- \partial_{\tau_2} e_{z_2} - \partial_{\tau_1} e_{z_3} - \frac{1}{\epsilon_0} \partial_{x_1} p_{x_3} \\
&- \frac{1}{\epsilon_0} \partial_{y_1} p_{y_3} - \frac{1}{\epsilon_0} \partial_{\tau_3} p_{z_1} \\
&- \frac{1}{\epsilon_0} \partial_{\tau_2} p_{z_2} - \frac{1}{\epsilon_0} \partial \tau_1 p_{z_3} - \frac{1}{\epsilon_0} \partial_\theta p_{z_4}.
\end{aligned} \tag{394}$$

Now let's summarize what we have. We have the values (320), (322), (337), (355), (373), (391), (335), (353), (371) (389), the equations (340), (358), (376), (394), (316), (318), (336), (338), (354), (356), (372), (374), (390), (392), (339), (357), (375), (393), and the solvability conditions (331), (332), (333), (334), (349), (350), (351), (352), (367), (368), (369), (370), (385), (386), (387) and (388). They become the system of equations

$$b_{z_0} = 0, \tag{395}$$
$$e_{z_0} = 0, \tag{396}$$
$$e_{x_1} = 0, \tag{397}$$
$$e_{y_1} = 0, \tag{398}$$
$$e_{x_2} = 0, \tag{399}$$
$$e_{y_2} = 0, \tag{400}$$
$$e_{x_3} = 0, \tag{401}$$
$$e_{x_4} = 0, \tag{402}$$

the equations

$$b_{x_0} = -\frac{1}{c} e_{y_0}, \tag{403}$$

$$b_{y_0} = \frac{1}{c} e_{x_0}, \tag{404}$$

$$c\partial_\theta b_{x_1} = \partial_{y_1} e_{z_0} - \partial_{\tau_1} e_{y_0}, \tag{405}$$

$$-c\partial_\theta b_{y_1} = \partial_{x_1} e_{z_0} - \partial_{\tau_1} e_{x_0}, \tag{406}$$

$$c\partial_\theta b_{z_1} = \partial_{x_1} e_{y_0} - \partial_{y_1} e_{x_0}, \tag{407}$$

$$\partial_\theta e_{z_1} = -\partial_{x_1} e_{x_0} - \partial_{y_1} e_{y_0} - \partial_{\tau_1} e_{z_0} - \frac{1}{\epsilon_0}\partial_\theta p_{z_1}, \tag{408}$$

$$c\partial_\theta b_{x_2} = \partial_{y_1} e_{z_1} - \partial_{\tau_1} e_{y_1} - \partial_{\tau_2} e_{y_0}, \tag{409}$$

$$-c\partial_\theta b_{y_2} = \partial_{x_1} e_{z_1} - \partial_{\tau_1} e_{x_1} - \partial_{\tau_2} e_{x_0}, \tag{410}$$

$$c\partial_\theta b_{z_2} = \partial_{x_1} e_{y_1} - \partial_{y_1} e_{x_1}, \tag{411}$$

$$\begin{aligned}\partial_\theta e_{z_2} = &-\partial_{x_1} e_{x_1} - \partial_{y_1} e_{y_1} - \partial_{\tau_2} e_{z_0} - \partial_{\tau_1} e_{z_1} - \frac{1}{\epsilon_0}\partial_{x_1} p_{x_1} \\ &- \frac{1}{\epsilon_0}\partial_{y_1} p_{y_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_1} - \frac{1}{\epsilon_0}\partial_\theta p_{z_2}.\end{aligned} \tag{412}$$

$$c\partial_\theta b_{x_3} = \partial_{y_1} e_{z_2} - \partial_{\tau_1} e_{y_2} - \partial_{\tau_2} e_{y_1} - \partial_{\tau_3} e_{y_0}, \tag{413}$$

$$-c\partial_\theta b_{y_3} = \partial_{x_1} e_{z_2} - \partial_{\tau_1} e_{x_2} - \partial_{\tau_2} e_{x_1} - \partial_{\tau_3} e_{x_0}, \tag{414}$$

$$c\partial_\theta b_{z_3} = \partial_{x_1} e_{y_2} - \partial_{y_1} e_{x_2}, \tag{415}$$

$$\begin{aligned}\partial_\theta e_{z_3} = &-\partial_{x_1} e_{x_2} - \partial_{y_1} e_{y_2} - \partial_{\tau_3} e_{z_0} - \partial_{\tau_2} e_{z_1} - \partial_{\tau_1} e_{z_2} \\ &- \frac{1}{\epsilon_0}\partial_{x_1} p_{x_2} - \frac{1}{\epsilon_0}\partial_{y_1} p_{y_2} - \frac{1}{\epsilon_0}\partial_{\tau_2} p_{z_1} - \frac{1}{\epsilon_0}\partial_{\tau_1} p_{z_2} \\ &- \frac{1}{\epsilon_0}\partial_\theta p_{z_3},\end{aligned} \tag{416}$$

$$c\partial_\theta b_{x_4} = \partial_{y_1} e_{z_3} - \partial_{\tau_1} e_{y_3} - \partial_{\tau_2} e_{y_2} - \partial_{\tau_3} e_{y_1} - \partial_{\tau_4} e_{y_0}, \tag{417}$$

$$-c\partial_\theta b_{y_4} = \partial_{x_1} e_{z_3} - \partial_{\tau_1} e_{x_3} - \partial_{\tau_2} e_{x_2} - \partial_{\tau_3} e_{x_1} - \partial_{\tau_4} e_{x_0}, \tag{418}$$

$$c\partial_\theta b_{z_4} = \partial_{x_1} e_{y_3} - \partial_{y_1} e_{x_3}, \tag{419}$$

$$\begin{aligned}\partial_\theta e_{z_4} = &-\partial_{x_1} e_{x_3} - \partial_{y_1} e_{y_3} - \partial_{\tau_4} e_{z_0} - \partial_{\tau_3} e_{z_1} \\ &- \partial_{\tau_2} e_{z_2} - \partial_{\tau_1} e_{z_3} - \frac{1}{\epsilon_0}\partial_{x_1} p_{x_3} \\ &- \frac{1}{\epsilon_0}\partial_{y_1} p_{y_3} - \frac{1}{\epsilon_0}\partial_{\tau_3} p_{z_1} \\ &- \frac{1}{\epsilon_0}\partial_{\tau_2} p_{z_2} - \frac{1}{\epsilon_0}\partial\tau_1 p_{z_3} - \frac{1}{\epsilon_0}\partial_\theta p_{z_4},\end{aligned} \tag{420}$$

and the solvability conditions

$$\partial_{y_1} e_{z_0} - \partial_{\tau_1} e_{y_0} = c(\partial_{x_1} b_{z_0} - \partial_{\tau_1} b_{x_0} - \mu_0 c \partial_\theta p_{y_1}), \tag{421}$$

$$\partial_{x_1} e_{z_0} - \partial_{\tau_1} e_{x_0} = c(\partial_{y_1} b_{z_0} - \partial_{\tau_1} b_{y_0} + \mu_0 c \partial_\theta p_{x_1}), \tag{422}$$

$$\partial_{x_1} e_{y_0} - \partial_{y_1} e_{x_0} = c(-\partial_{x_1} b_{x_0} - \partial_{y_1} b_{y_0} - \partial_{\tau_1} b_{z_0}), \tag{423}$$

$$\begin{aligned} &- \partial_{x_1} e_{x_0} - \partial_{y_1} e_{y_0} - \partial_{\tau_1} e_{z_0} - \frac{1}{\epsilon_0} \partial_\theta p_{z_1} = c(\partial_{y_1} b_{x_0} - \partial_{x_1} b_{y_0} \\ &- \mu_0 c \partial_\theta p_{z_1}), \end{aligned} \tag{424}$$

$$\partial_{y_1} e_{z_1} - \partial_{\tau_1} e_{y_1} - \partial_{\tau_2} e_{y_0} = c(\partial_{x_1} b_{z_1} - \partial_{\tau_1} b_{x_1} - \partial_{\tau_2} b_{x_0} + \mu_0 c \partial_\theta p_{y_2}), \tag{425}$$

$$\partial_{x_1} e_{y_1} - \partial_{y_1} e_{x_1} = c(-\partial_{x_1} b_{x_1} - \partial_{y_1} b_{y_1} - \partial_{\tau_1} b_{z_1} - \partial_{\tau_2} b_{z_0}), \tag{426}$$

$$\begin{aligned} &\partial_{x_1} e_{x_1} - \partial_{y_1} e_{y_1} - \partial_{\tau_2} e_{z_0} - \partial_{\tau_1} e_{z_1} \\ &- \frac{1}{\epsilon_0} \partial_{x_1} p_{x_1} - \frac{1}{\epsilon_0} \partial_{y_1} p_{y_1} - \frac{1}{\epsilon_0} \partial_{\tau_1} p_{z_1} - \frac{1}{\epsilon_0} \partial_\theta p_{z_2} \\ &= c(\partial_{y_1} b_{x_1} - \partial_{x_1} b_{y_1} - \mu_0 c \partial_\theta p_{z_2}). \end{aligned} \tag{427}$$

$$\partial_{x_1} e_{z_1} - \partial_{\tau_1} e_{x_1} - \partial_{\tau_2} e_{x_0} = c(\partial_{y_1} b_{z_1} - \partial_{\tau_1} b_{y_1} - \partial_{\tau_2} b_{y_0} + \mu_0 c \partial_\theta p_{x_2}), \tag{428}$$

$$\begin{aligned} &\partial_{y_1} e_{z_2} - \partial_{\tau_1} e_{y_2} - \partial_{\tau_2} e_{y_1} - \partial_{\tau_3} e_{y_0} \\ &= c(\partial_{x_1} b_{z_2} - \partial_{\tau_1} b_{x_2} - \partial_{\tau_2} b_{x_1} - \partial_{\tau_3} b_{x_0} - \mu_0 c \partial_\theta p_{y_3}), \end{aligned} \tag{429}$$

$$\begin{aligned} &\partial_{x_1} e_{z_2} - \partial_{\tau_1} e_{x_2} - \partial_{\tau_2} e_{x_1} - \partial_{\tau_3} e_{x_0} \\ &= \partial_{y_1} b_{z_2} - \partial_{\tau_1} b_{y_2} - \partial_{\tau_2} b_{y_1} - \partial_{\tau_3} b_{y_0} + \mu_0 c \partial_\theta p_{x_3}, \end{aligned} \tag{430}$$

$$\partial_{x_1} e_{y_2} - \partial_{y_1} e_{x_2} = c(-\partial_{x_1} b_{x_2} - \partial_{y_1} b_{y_2} - \partial_{\tau_1} b_{z_2} - \partial_{\tau_2} b_{z_1} - \partial_{\tau_3} b_{z_0}), \tag{431}$$

$$\begin{aligned} &- \partial_{x_1} e_{x_2} - \partial_{y_1} e_{y_2} - \partial_{\tau_3} e_{z_0} - \partial_{\tau_2} e_{z_1} - \partial_{\tau_1} e_{z_2} \\ &- \frac{1}{\epsilon_0} \partial_{x_1} p_{x_2} - \frac{1}{\epsilon_0} \partial_{y_1} p_{y_2} - \frac{1}{\epsilon_0} \partial_{\tau_2} p_{z_1} - \frac{1}{\epsilon_0} \partial_{\tau_1} p_{z_2} \\ &- \frac{1}{\epsilon_0} \partial_\theta p_{z_3} = c(\partial_{y_1} b_{x_2} - \partial_{x_1} b_{y_2} - \mu_0 c \partial_\theta p_{z_3}), \end{aligned} \tag{432}$$

$$\begin{aligned} &\partial_{y_1} e_{z_3} - \partial_{\tau_1} e_{y_3} - \partial_{\tau_2} e_{y_2} - \partial_{\tau_3} e_{y_1} - \partial_{\tau_4} e_{y_0} \\ &= c(\partial_{x_1} b_{z_3} - \partial_{\tau_1} b_{x_3} - \partial_{\tau_2} b_{x_2} - \partial_{\tau_3} b_{x_1} - \partial_{\tau_4} b_{x_0} - \mu_0 c \partial_\theta p_{y_4}), \end{aligned} \tag{433}$$

$$\begin{aligned} &\partial_{x_1} e_{z_3} - \partial_{\tau_1} e_{x_3} - \partial_{\tau_2} e_{x_2} - \partial_{\tau_3} e_{x_1} - \partial_{\tau_4} e_{x_0} \\ &= c(\partial_{y_1} b_{z_3} - \partial_{\tau_1} b_{y_3} - \partial_{\tau_2} b_{y_2} - \partial_{\tau_3} b_{y_1} \\ &- \partial_{\tau_4} b_{y_0} + \mu_0 c \partial_\theta p_{x_4}), \end{aligned} \tag{434}$$

$$\begin{aligned} &\partial_{x_1} e_{y_3} - \partial_{y_1} e_{x_3} \\ &= c(-\partial_{x_1} b_{x_3} - \partial_{y_1} b_{y_3} - \partial_{\tau_1} b_{z_3} - \partial_{\tau_2} b_{z_2} - \partial_{\tau_3} b_{z_1} - \partial_{\tau_4} b_{z_0}), \end{aligned} \tag{435}$$

$$\begin{aligned} &- \partial_{x_1} e_{x_3} - \partial_{y_1} e_{y_3} - \partial_{\tau_4} e_{z_0} - \partial_{\tau_3} e_{z_1} \\ &- \partial_{\tau_2} e_{z_2} - \partial_{\tau_1} e_{z_3} - \frac{1}{\epsilon_0} \partial_{x_1} p_{x_3} \\ &- \frac{1}{\epsilon_0} \partial_{y_1} p_{y_3} - \frac{1}{\epsilon_0} \partial_{\tau_3} p_{z_1} \\ &- \frac{1}{\epsilon_0} \partial_{\tau_2} p_{z_2} - \frac{1}{\epsilon_0} \partial \tau_1 p_{z_3} - \frac{1}{\epsilon_0} \partial_\theta p_{z_4} \\ &= c(\partial_{y_1} b_{x_3} - \partial_{x_1} b_{y_3} - \mu_0 c \partial_\theta p_{z_4}). \end{aligned} \tag{436}$$

We are now going to use the equations (395) - (420) on the solution conditions. When finding the derivative on θ and simplify using (395)-(418) on the solution conditions (423), (424), (426), (427), (431), (432), (435) and (436). Simplifying them in this way shows that they are automatically satisfied.

Using (395)-(420) on the rest of the solvability conditions in the same way as in the previous chapters let us end up with the equations

$$2\partial_{\tau_1\theta}e_{x_0} = \mu_0 c^2 \partial_{\theta\theta} p_{x_1}, \tag{437}$$

$$2\partial_{\tau_1\theta}e_{y_0} = \mu_0 c^2 \partial_{\theta\theta} p_{y_1}, \tag{438}$$

$$\begin{aligned} 2\partial_{\tau_2\theta}e_{y_0} &= -\partial_{y_1y_1}e_{y_0} - \frac{1}{\epsilon_0}\partial_{\theta y_1}p_{z_1} - \partial_{x_1x_1}e_{y_0} \\ &- \partial_{\tau_1\tau_1}e_{y_0} + \mu_0 c^2 \partial_{\theta\theta}p_{y_2}, \end{aligned} \tag{439}$$

$$\begin{aligned} 2\partial_{\tau_2\theta}e_{x_0} &= -\partial_{y_1y_1}e_{x_0} - \partial_{\tau_1\tau_1}e_{x_0} - \partial_{x_1x_1}e_{x_0} \\ &- \frac{1}{\epsilon_0}\partial_{x_1\theta}p_{z_1} + \mu_0 c^2 \partial_{\theta\theta}p_{x_2}, \end{aligned} \tag{440}$$

$$\begin{aligned} 2\partial_{\tau_3\theta}e_{y_0} &= -2\partial_{\tau_1\tau_2}e_{y_0} - \frac{1}{\epsilon_0}(\partial_{y_1x_1}p_{x_1} + \partial_{y_1y_1}p_{y_1} + \partial_{y_1\tau_1}p_{z_1} \\ &+ \partial_{y_1\theta}p_{z_2}) + \mu_0 c^2 \partial_{\theta\theta}p_{y_3}, \end{aligned} \tag{441}$$

$$\begin{aligned} 2\partial_{\tau_3\theta}e_{x_0} &= -2\partial_{\tau_1\tau_2}e_{x_0} + \mu_0 c^2 \partial_{\theta\theta}p_{x_3} \\ &- \frac{1}{\epsilon_0}(\partial_{x_1x_1}p_{x_1} + \partial_{x_1y_1}p_{y_1} + \partial_{x_1\tau_1}p_{z_1} + \partial_{x_1\theta}p_{z_2}), \end{aligned} \tag{442}$$

$$\begin{aligned} 2\partial_{\tau_4\theta}e_{y_0} &= -\frac{1}{\epsilon_0}(\partial_{y_1x_1}p_{x_2} + \partial_{y_1y_1}p_{y_2} + \partial_{\tau_2y_1}p_{z_1} + \partial_{y_1\tau_1}p_{z_2} \\ &+ \partial_{y_1\theta}p_{z_3}) - 2\partial_{\tau_1\tau_3}e_{y_0} - \partial_{\tau_2\tau_2}e_{y_0} + \mu_0 c^2 \partial_{\theta\theta}p_{y_4}, \end{aligned} \tag{443}$$

$$\begin{aligned} 2\partial_{\tau_4\theta}e_{x_0} &= -\frac{1}{\epsilon_0}(\partial_{x_1x_1}p_{x_2} + \partial_{x_1y_1}p_{y_2} + \partial_{x_1\tau_2}p_{z_1} \\ &+ \partial_{x_1\tau_1}p_{z_2} + \partial_{x_1\theta}p_{z_3}) - 2\partial_{\tau_1\tau_3}e_{x_0} - 2\partial_{\tau_2\tau_2}e_{y_0} + \mu_0 c^2 \partial_{\theta\theta}p_{x_4}. \end{aligned} \tag{444}$$

Consider the special case when

$$P = \epsilon^2 \left(\epsilon_0 \hat{\chi}(-ic\partial_\theta)e_0 + \epsilon_0 \eta e_0^3\right),$$

which implies that

$$\begin{aligned} p_1 &= 0, \\ p_2 &= \left(\epsilon_0 \hat{\chi}(-ic\partial_\theta)e_0 + \epsilon_0 \eta e_0^3\right), \\ p_3 &= 0, \\ p_4 &= 0. \end{aligned}$$

Introduce

$$E_{x_0}(\theta, x, \tau_1, \tau_2, ...)_{x_1=\epsilon x, \tau_j=\epsilon^j \tau}. \tag{445}$$

$$E_{y_0}(\theta, x, \tau_1, \tau_2, ...)_{x_1=\epsilon x, \tau_j=\epsilon^j \tau} \tag{446}$$

Multiplying (437) and (438) by ϵ, (439) and (440) by ϵ^2, (441) and (442) by ϵ^3 and (443) and (444) by ϵ^4, and using the expansions

$$\partial_x = \epsilon \partial_{x_1}, \tag{447}$$

$$\partial_\tau = \epsilon \partial_{\tau_1} + \epsilon^2 \partial_{\tau_2} + \epsilon^3 \partial_{\tau_3} + \epsilon^4 \partial_{\tau_4}, \tag{448}$$

we get

$$\begin{aligned} 2\partial_{\tau\theta} E_{x_0} = & -\partial_{yy} E_{x_0} - \partial_{xx} E_{x_0} - \partial_{\tau\tau} E_{x_0} \\ & - (\epsilon^2 \partial_{xx} (\hat{\chi}(-ic\partial_\theta) E_{x_0} + \eta(E_{x_0}^3 + E_{y_0}^2 E_{x_0})) \\ & + \epsilon^2 \partial_{xy} (\epsilon_0 \hat{\chi}(-ic\partial_\theta) E_{y_0} + \epsilon_0 \eta(E_{x_0}^2 E_{y_0} + E_{y_0}^3)) + \\ & \epsilon^3 \partial_{x\theta} \epsilon_0 \eta(E_{x_0}^2 + E_{y_0}^2) E_{z_1}) \\ & + \mu_0 \epsilon^2 c^2 \partial_{\theta\theta} (\epsilon_0 \hat{\chi}(-ic\partial_\theta) E_{x_0} + \epsilon_0 \eta(E_{x_0}^3 + E_{y_0}^2 E_{x_0})), \end{aligned} \tag{449}$$

$$\begin{aligned} 2\partial_{\tau\theta} E_{y_0} = & -\partial_{yy} E_{y_0} - \partial_{xx} E_{y_0} - \partial_{\tau\tau} E_{y_0} \\ & - (\epsilon^2 \partial_{xy} (\hat{\chi}(-ic\partial_\theta) E_{x_0} + \eta(E_{x_0}^3 + E_{y_0}^2 E_{x_0})) + \\ & \epsilon^2 \partial_{yy} (\epsilon_0 \hat{\chi}(-ic\partial_\theta) E_{y_0} + \eta\epsilon_0 (E_{x_0}^2 E_{y_0} + E_{y_0}^3)) \\ & + \epsilon^3 \partial_{y\theta} \epsilon_0 \eta(E_{x_0}^2 + E_{y_0}^2) E_{z_1}) + \mu_0 \epsilon_0 c^2 \epsilon^2 \partial_{\theta\theta} (\hat{\chi}(-ic\partial_\theta) E_{y_0} \\ & + \eta(E_{x_0}^2 E_{y_0} + E_{y_0}^3)). \end{aligned} \tag{450}$$

5 Perturbation equations to order ϵ^2 without polarization

We now have equations (449) and (450)

$$\begin{aligned}2\partial_{\tau\theta}E_{x_0} = &-\partial_{yy}E_{x_0} - \partial_{xx}E_{x_0} - \partial_{\tau\tau}E_{x_0}\\ &- (\epsilon^2\partial_{xx}(\hat{\chi}(-ic\partial_\theta)E_{x_0} + \eta(E_{x_0}^3 + E_{y_0}^2E_{x_0}))\\ &+ \epsilon^2\partial_{xy}(\epsilon_0\hat{\chi}(-ic\partial_\theta)E_{y_0} + \epsilon_0\eta(E_{x_0}^2E_{y_0} + E_{y_0}^3))+\\ &\epsilon^3\partial_{x\theta}\epsilon_0\eta(E_{x_0}^2 + E_{y_0}^2)E_{z_1})\\ &+ \mu_0\epsilon^2c^2\partial_{\theta\theta}(\epsilon_0\hat{\chi}(-ic\partial_\theta)E_{x_0} + \epsilon_0\eta(E_{x_0}^3 + E_{y_0}^2E_{x_0})),\end{aligned}$$

$$\begin{aligned}2\partial_{\tau\theta}E_{y_0} = &-\partial_{yy}E_{y_0} - \partial_{xx}E_{y_0} - \partial_{\tau\tau}E_{y_0}\\ &- (\epsilon^2\partial_{xy}(\hat{\chi}(-ic\partial_\theta)E_{x_0} + \eta(E_{x_0}^3 + E_{y_0}^2E_{x_0}))+\\ &\epsilon^2\partial_{yy}(\epsilon_0\hat{\chi}(-ic\partial_\theta)E_{y_0} + \eta\epsilon_0(E_{x_0}^2E_{y_0} + E_{y_0}^3))\\ &+ \epsilon^3\partial_{y\theta}\epsilon_0\eta(E_{x_0}^2 + E_{y_0}^2)E_{z_1}) + \mu_0\epsilon_0c^2\epsilon^2\partial_{\theta\theta}(\hat{\chi}(-ic\partial_\theta)E_{y_0}\\ &+ \eta(E_{x_0}^2E_{y_0} + E_{y_0}^3)).\end{aligned}$$

Taking away the dispersion and the diffraction from (449) and (450) and dropping all terms of order ϵ^3 or higher we get

$$2\partial_{\theta\tau}E_{x_0} = \mu_0\epsilon_0c^2\epsilon^2\partial_{\theta\theta}\eta(E_{x_0}^3 + E_{x_0}E_{y_0}^2), \tag{451}$$

$$2\partial_{\theta\tau}E_{y_0} = \mu_0\epsilon_0c^2\epsilon^2\partial_{\theta\theta}\eta(E_{x_0}^2E_{y_0} + E_{y_0}^3). \tag{452}$$

Integrating over θ on both sides will give:

$$2\partial_\tau E_x = \mu_0\epsilon_0c^2\epsilon^2\partial_\theta\eta(E_{x_0}^3 + E_{x_0}E_{y_0}^2) + f(\tau), \tag{453}$$

$$2\partial_\tau E_y = \mu_0\epsilon_0c^2\epsilon^2\partial_\theta\eta(E_{x_0}^2e_{y_0} + E_{y_0}^3) + g(\tau). \tag{454}$$

We are choosing to disregard $f(\tau)$ and $g(\tau)$. Doing a change of variables where $\tau = \tau_0\tau'$ and $\theta = \theta_0\theta'$ and choosing $\theta_0 = \mu_0\epsilon_0c^2\epsilon^2\eta\tau_0$ will give the equations:

$$2\partial_\tau E_{x_0} = \partial_\theta(E_{x_0}^3 + e_{x_0}E_{y_0}^2), \tag{455}$$

$$2\partial_\tau E_{y_0} = \partial_\theta(E_{x_0}^2e_{y_0} + E_{y_0}^3). \tag{456}$$

Where we have returned to unprimed quantities the case of linear polarization where $E_{y_0} = 0$. Equation (456) will then disappear and equation (455) will become:

$$2\partial_\tau E_{x_0} = \partial_\theta(E_{x_0}^3) = 3E_{x_0}^2 \partial E_{x_0}. \tag{457}$$

This is now a quasilinear first order partial differential equation. This is because it is linear in the derivative terms, but has a nonlinear expression $3E_{x_0}^2 \partial_\theta E_{x_0}$.

$$2\partial_\tau E_{x_0} - 3E_{x_0}^2 \partial_\theta E_{x_0} = 0. \tag{458}$$

With an initial value $E_{x_0}(\theta, 0) = f(\theta)$. It can be solved using the method of characteristics[6]. First parameterize the initial curve

$$\theta = t \qquad \tau = 0 \qquad E_{x_0} = f(t), \tag{459}$$

and find the value of

$$J = \frac{\partial \tau}{\partial t}(-3E_{x_0}^2) - \frac{\partial \theta}{\partial t}(2) = -2 \neq 0. \tag{460}$$

This means that there exists one and only one solution to this equation. It's necessary to find the motion of the wave, and that means finding the velocity of $\frac{\partial \theta}{\partial \tau}$ of each point of the wave. The first two parts of the characteristic equations (423) means that, the bigger the amplitude $|E_{x_0}(\theta, \tau)|$ of the wave is, the bigger is the speed of the corresponding point of the wave θ.

$$\frac{\partial \theta}{\partial s} = -3E_{x_0}^2 \qquad \frac{\partial \tau}{\partial s} = 2 \qquad \frac{\partial E_{x_0}}{\partial s} = 0. \tag{461}$$

with the initial condition $s = 0$.

$$\frac{\partial \theta}{\partial s} = -3E_{x_0}^2,$$

$$\implies \theta = -3E_{x_0}^2 s + t = -3f(t)^2 s + t, \tag{462}$$

$$\frac{\partial \tau}{\partial s} = 2,$$

$$\implies \tau = 2s + c = 2s. \tag{463}$$

$\frac{\partial E_{x_0}}{\partial s} = 0$ means that $E_{x_0}(s, t)$ is constant along the characteristic curves such that $E_{x_0}(s, t) = E_{x_0}(0, t) = f(t)$.

$$\theta = -\frac{3}{2}f(t)^2\tau + t, \tag{464}$$

$$\implies t = \theta + \frac{3}{2}f(t)^2\tau, \tag{465}$$

$$= \theta + \frac{3}{2}E_{x_0}^2\tau, \tag{466}$$

$$E_{x_0} = f(t(\theta, \tau)). \tag{467}$$

The implicit solution is

$$E_{x_0} = f(\theta + \frac{3}{2}E_{x_0}^2\tau). \tag{468}$$

5.1 Breaking Time

If $E_{x_0} > 0$ the point θ will move to the right, if $E_{x_0} = 0$ θ will be fixed, and if $E_{x_0} < 0$ will θ move to the left. But if $E_{x_0}(\theta, \tau)$ takes on both positive and negative values, different parts of the wave will move with different speeds to the right or to the left. In this case the wave will move to the left. That means that the points θ where E_{x_0} has higher values will move faster to the left than the points where E_{x_0} has smaller values. If the higher parts of the wave form initially are to the right or the rear of the of lower parts, then will the higher parts eventually pass the lower parts. The first time this happens is when the wave breaks, and E_{x_0} becomes multivalued and is no longer a valid solution. This means that equation (458) no longer is an acceptable model for the physical process, and the neglected parts of the quasilinear equation (458) are significant. It's possible to use implicit derivation to find both the time τ and the point θ where the wave breaks:

$$\partial_\theta E_{x_0} = f'(\theta + \frac{3}{2}E_{x_0}^2\tau)(1 + \frac{3}{2}\tau \cdot 2E_{x_0}\partial_\theta E_{x_0}), \tag{469}$$

$$= f'(\theta + \frac{3}{2}E_{x_0}^2\tau)(1 + 3\tau E_{x_0}\partial_\theta E_{x_0}), \tag{470}$$

$$= f'(\theta\frac{3}{2}E_{x_0}^2\tau) + 3\tau E_{x_0}\partial_\theta E_{x_0} f'(\theta + \frac{3}{2}E_{x_0}^2\tau), \tag{471}$$

$$\implies \partial_\theta E_{x_0}(1 - 3\tau E_{x_0} f'(\theta + \frac{3}{2}E_{x_0}^2\tau)) = f'(\theta + \frac{3}{2}E_{x_0}^2\tau), \tag{472}$$

$$\implies \partial_\theta E_{x_0} = \frac{f'(\theta + \frac{3}{2}E_{x_0}^2\tau)}{1 - 3\tau E_{x_0} f'(\theta + \frac{3}{2}E_{x_0}^2\tau)}, \tag{473}$$

$$= \frac{f'(t)}{1 - 3\tau f(t)f'(t)}. \tag{474}$$

The equation breaks down when $1 - 3\tau f(t)f'(t) = 0$:

$$3\tau f(t)f'(t) = 1, \tag{475}$$

$$\implies \tau = \frac{1}{3f(t)f'(t)}. \tag{476}$$

The breakdown time will then be:

$$\tau* = \min_t \left(\frac{1}{3f(t)f'(t)} \right). \tag{477}$$

5.2 Numerical solution

The finite difference method[2] is a numerical method that is used to find the numerical solution of ordinary and partial differential equations. The method solves equations by discretization of the equations on the space-time grid in figure 3

That means that the equation $E_{x_0}(\theta, \tau)$ becomes $E_{x_0}(\theta_i, \tau_n)$. The finite difference method use Taylors theorem to find an approximation to the derivatives of the equations expressed by the points on the space- time grid. Using this gives the forward difference

$$\left(\frac{\partial E_{x_0}}{\partial \tau} \right)_i^n \approx \frac{(E_{x_0})_i^{n+1} - (E_{x_0})_i^n}{d\tau}, \tag{478}$$

backward difference

$$\left(\frac{\partial E_{x_0}}{\partial \tau} \right)_i^n \approx \frac{(E_{x_0})_i^{n-1} - (E_{x_0})_i^n}{d\tau}, \tag{479}$$

and center difference

$$\left(\frac{\partial E_{x_0}}{\partial \tau} \right)_i^n \approx \frac{(E_{x_0})_i^{n+1} - (E_{x_0})_i^{n-1}}{2d\tau}. \tag{480}$$

Center difference has an approximate error of $d\tau^2$, while forward and backward difference has an approximate error of $d\tau$. This means that center difference has a more accurate approximation, something that makes it the best choice in this case to use to find the numerical solution to equation (458). Choosing in this case to use periodic boundary conditions, which means that the first point and the last point on each line in the space- time

grid have the same value. This is something that is going to be used when finding the equations that's used in the code.

The first line for when n=0 is given as a Gaussian function. The equations for center difference uses points from the two last lines to find a point on the next line. It is therefore impossible to use center difference to find the points for n=1, and we have to use forward difference for the derivative on τ, and the equation to find the points (not the end points) when n=1 is

$$(E_{x_0})_i^1 = (E_{x_0})_i^0 + \frac{3}{4}s((E_{x_0})_i^0)^2((E_{x_0})_{i+1}^0 - (E_{x_0})_{i-1}^0), \tag{481}$$

where $s = \frac{d\tau}{d\theta}$. Because of the boundary conditions will the equation for the boundary points need to be different from equation (433) with the derivative on θ. The equation of i=0 is

$$(E_{x_0})_0^1 = (E_{x_0})_0^0 + \frac{3}{4}s((E_{x_0})_0^0)^2((E_{x_0})_1^0 - (E_{x_0})_{N-2}^0), \tag{482}$$

and the equation for the last point when i=N-2 is

$$(E_{x_0})_{N-2}^1 = (E_{x_0})_{N-2}^0 + \frac{3}{4}s((E_{x_0})_{N-2}^0)^2((E_{x_0})_0^0 - (E_{x_0})_{N-3}^0), \tag{483}$$

where $s = \frac{d\tau}{d\theta}$. The center difference equation in general except the boundary points is

$$(E_{x_0})_i^{n+1} = (E_{x_0})_i^{n-1} + \frac{3}{2}s((E_{x_0})_i^n)^2((E_{x_0})_{i+1}^n - (E_{x_0})_{i-1}^n). \tag{484}$$

The center difference equation for the boundary point i=0 is

$$(E_{x_0})_0^{n+1} = (E_{x_0})_0^{n-1} + \frac{3}{2}s((E_{x_0})_0^n)^2((E_{x_0})_1^n - (E_{x_0})_{N-2}^n), \tag{485}$$

and the center difference equation for the boundary point i=N-2 is

$$(E_{x_0})_{N-2}^{n+1} = (E_{x_0})_{N-2}^{n-1} + \frac{3}{2}s((E_{x_0})_{N-2}^n)^2((E_{x_0})_0^n - (E_{x_0})_{N-3}^n). \tag{486}$$

5.2.1 Initial function

The numerical solution uses an initial function, which in this case is the initial laser pulse. This laser pulse has the form

$$E_{x_0}(0,t) = f(t)\cos(\omega_0 t). \tag{487}$$

Where $f(t)$ is the pulse shape function, which can also be called an envelope function. The common choice for this type of function is a Gaussian, which has the form

$$f(t) = a\exp(-bx^2), \tag{488}$$

which is a function symmetrical around t =0. A typical shape for this is shown in figure 4, where the period is

$$T = \frac{2\pi}{\omega_0}. \tag{489}$$

The oscillating part of $E_{x_0}(0,t)$ is called the "carrier" wave. Introducing the change of variables (37) and (38) to equation (496) will give the initial function in the(θ,τ) plane

$$A(\theta,0) = f\left(-\frac{\theta}{c}\right)\cos\left(-\omega_0\frac{\theta}{c}\right). \tag{490}$$

Inserting (526) into (528) and using the scaling

$$A = \alpha_0 A' \tag{491}$$

$$\theta = \theta_0\theta' \tag{492}$$

will give the function

$$A'(\theta',0) = \frac{a}{\alpha}\exp\left(-\frac{b}{c^2}\theta_0^2\theta'^2\right)\cos\left(\frac{\omega_0\theta_0}{c}\theta'\right). \tag{493}$$

Choosing $a = \alpha$ so the amplitude is normalized to one, and choosing the scaling of θ_0 such that the carrier wave as a period of 2π is

$$\frac{\omega_0\theta_0}{c} = 1. \tag{494}$$

Equation (493) now becomes

$$A'(\theta',0) = \exp(-\gamma\theta'^2)\cos(\theta'), \tag{495}$$

where $\gamma = \frac{b}{\omega_0^2}$.

Since b is free,then γ is a free dimensionless number, and by varying the number γ can we get as few or as many oscillations in the carrier wave under the envelope wave as we want.

5.2.2 Stability

The numerical solution, with in this case a chosen Gaussian initial function, can look like shown in figure 5.

This is because of a numerical instability in our finite difference method. To find the stability condition let's look at the linear case of equation (458), which is

$$\partial_\tau E_{x_0} + c\partial_\theta E_{x_0} = 0. \tag{496}$$

It's now possible to use this equation to find a stability condition using separation of variables on the center difference equation for equation (496)

$$(E_{x_0})_j^n = (E_{x_0})(x_j, t_n), \tag{497}$$

and the center difference equation for (496) is

$$\frac{(E_{x_0})_j^{n+1} - (E_{x_0})_j^{n-1}}{2dt} + c\frac{(E_{x_0})_{j+1}^n - (E_{x_0})_{j-1}^n}{2dx} = 0,$$

$$\implies (E_{x_0})_j^{n+1} - (E_{x_0})_j^{n-1} + cs((E_{x_0})_{j+1}^n - (E_{x_0})_{j-1}^n) = 0, \tag{498}$$

where $s = c\frac{dt}{dx}$. Now let's assume that

$$(E_{x_0})_j^n = \xi^n \eta_j \tag{499}$$

Inserting (528) into equation (528) will give

$$\xi^{n+1}\eta_j = \xi^{n-1}\eta_j + s(\xi^n\eta_{j+1} - \xi^n\eta_{j-1}, \tag{500}$$

$$\implies \xi^{n+1} = \xi^{n-1} + s\xi^n(\eta_1 - \eta_{-1}). \tag{501}$$

Still assuming periodic boundary conditions

$$\eta_0 = \eta_{N-1}, \tag{502}$$

such that

$$(E_{x_0})_j^n = \xi^n \exp(i\frac{2\pi j}{N-1}) = \xi^n \exp(i\theta). \tag{503}$$

Where for large N θ_j is approximated by a continuous variable θ. This means that equation (501) becomes

$$\xi^{n+1} = \xi^{n-1} + \xi^n s(\exp(i\theta) - \exp(-i\theta)). \tag{504}$$

Using the fact that

$$(\exp(i\theta) - \exp(-i\theta)) = 2i\sin(\theta) \tag{505}$$

equation (504) will become

$$\xi^1 = \xi^{-1} + 2si\sin(\theta). \tag{506}$$

This becomes the second order polynomial equation

$$\xi^2 - 2is\sin(\theta)\xi - 1 = 0. \tag{507}$$

Equation (507) has the solution

$$\xi = \frac{1}{2}\left(2si\sin(\theta) \pm \sqrt{-4s^2\sin^2(\theta) + 4}\right). \tag{508}$$

when assuming that $s < 1$ the norm of ξ will be

$$|\xi|^2 = s^2\sin^2(\theta) + 1 - \sin^2(\theta) = 1. \tag{509}$$

Since the requirement of stability is that

$$|\xi| \leq 1 \tag{510}$$

can we say that the numerical scheme (498) is stable for $s < 1$.

When $s > 1$

$$s^2\sin^2(\theta) - 1 > 0, \tag{511}$$

which means that

$$|\xi|^2 > 1, \tag{512}$$

and the numerical scheme (498) is not stable for $s > 1$.

This is for the linear case, but the quasilinear case that we have

$$c = c(E_{x_0}) = \frac{3}{2}E_{x_0}^2, \tag{513}$$

the requirement for stability is conjectured to be, based on the arguments given on the previous page

$$c(E_{x_0})s < 1, \tag{514}$$

$$\implies \frac{3}{2}E_{x_0}^2 s < 1 \tag{515}$$

which means that the numerical solution for equation (458) is stable if

$$s < \frac{2}{3(E_{x_0})_{max}^2}. \tag{516}$$

This makes sense since in light of figure 5 the instability appears on the top of the graph.

5.2.3 Testing

When choosing a Gaussian function as the initial function for the numerical solution is it possible to use equation (439) to test the implementation. This is done by finding the breaking time by inserting the initial function into equation (439), and then see if the time fits with the breaking time of the numerical solution. The starting function has the form

$$f(x) = a\exp(-\gamma x^2), \tag{517}$$

which has the derivative

$$f'(x) = -2\gamma x\exp(-\gamma x^2). \tag{518}$$

Inserting this into equation (438) defines the function $g(x)$

$$g(x) = -\frac{\exp(2\gamma x^2)}{6\gamma x}. \tag{519}$$

The minimum value of x is found by setting the derivative of $g(x) equal to zero$. We have

$$g'(x) = (-\frac{2}{3} + \frac{1}{6\gamma x})\exp(2\gamma x^2) = 0. \tag{520}$$

The solution to this equation is

$$x = \pm \frac{1}{2\gamma^{1/2}}. \tag{521}$$

The value of x that will give a positive time $\tau*$ is

$$x = -\frac{1}{2\gamma^{1/2}} \tag{522}$$

Inserting this into equation (519) give the equation

$$\tau* = \frac{\exp(1/2)}{3\gamma^{1/2}}. \tag{523}$$

Choosing a function where $\gamma = 0.01$ will give a breaking time

$$\tau* = 5,4957 \approx 5,5. \tag{524}$$

This corresponds to the numerical implementation, because we can see in figure 6 that the graph starts to break when $\tau = 5.5$.

5.3 Results

For the case when $\gamma = 0.01$ will the initial wave look like in figure 7.

Over time when the graph starts to lean to the left, because of the nonlinearity of function (458). This is shown in figure 8.

When it leans so much to the left that the graph has a vertical line, it will break. In this case it will break after 0.59 seconds.

When iterating a little bit more from the breaking point, it is shown in figure 10 that the function will no longer give valid solutions to the physical process.

6 Perturbation equations to order ϵ^4 without polarization

$$
\begin{aligned}
2\partial_{\tau\theta}E_{x_0} = &-\partial_{yy}E_{x_0} - \partial_{xx}E_{x_0} - \partial_{\tau\tau}E_{x_0} \\
&- (\epsilon^2\partial_{xx}(\hat{\chi}(-ic\partial_\theta)E_{x_0} + \eta(E_{x_0}^3 + E_{y_0}^2E_{x_0})) \\
&+ \epsilon^2\partial_{xy}(\epsilon_0\hat{\chi}(-ic\partial_\theta)E_{y_0} + \epsilon_0\eta(E_{x_0}^2E_{y_0} + E_{y_0}^3))+ \\
&\epsilon^3\partial_{x\theta}\epsilon_0\eta(E_{x_0}^2 + E_{y_0}^2)E_{z_1}) \\
&+ \mu_0\epsilon^2c^2\partial_{\theta\theta}(\epsilon_0\hat{\chi}(-ic\partial_\theta)E_{x_0} + \epsilon_0\eta(E_{x_0}^3 + E_{y_0}^2E_{x_0})),
\end{aligned}
$$

$$
\begin{aligned}
2\partial_{\tau\theta}E_{y_0} = &-\partial_{yy}E_{y_0} - \partial_{xx}E_{y_0} - \partial_{\tau\tau}E_{y_0} \\
&- (\epsilon^2\partial_{xy}(\hat{\chi}(-ic\partial_\theta)E_{x_0} + \eta(E_{x_0}^3 + E_{y_0}^2E_{x_0}))+ \\
&\epsilon^2\partial_{yy}(\epsilon_0\hat{\chi}(-ic\partial_\theta)E_{y_0} + \eta\epsilon_0(E_{x_0}^2E_{y_0} + E_{y_0}^3)) \\
&+ \epsilon^3\partial_{y\theta}\epsilon_0\eta(E_{x_0}^2 + E_{y_0}^2)E_{z_1}) + \mu_0\epsilon_0c^2\epsilon^2\partial_{\theta\theta}(\hat{\chi}(-ic\partial_\theta)E_{y_0} \\
&+ \eta(E_{x_0}^2E_{y_0} + E_{y_0}^3)).
\end{aligned}
$$

Removing the dispersion and the diffraction, but now retaining terms at order ϵ^4, we get the equations (410) and (411):

$$
\begin{aligned}
2\partial_{\theta\tau}E_x &= -\partial_{\tau\tau}E_{x_0} + \mu_0\epsilon_0c^2\epsilon^2\partial_{\theta\theta}\eta(E_{x_0}^3 + E_{x_0}E_{y_0}^2), \\
2\partial_{\theta\tau}E_y &= -\partial_{\tau\tau}E_{y_0} + \mu_0\epsilon_0c^2\epsilon^2\partial_{\theta\theta}\eta(E_{x_0}^2E_{y_0} + E_{y_0}^3).
\end{aligned}
$$

In this case will $\partial_{\tau\tau}$ not be disregarded, and the case of $E_{y_0} = 0$ is still valid. This makes the equation

$$
2\partial_{\theta\tau}E_{x_0} = -\partial_{\tau\tau}E_{x_0} + \mu_0c^2\epsilon^2\partial_{\theta\theta}E_{x_0}^3. \tag{525}
$$

Doing a change of variables where $\tau = \tau_0\tau'$ and $\theta = \theta_0 + \theta'$ and choosing $\theta_0 = \mu_0\epsilon_0c^2\epsilon^2\eta\tau_0$ will give the equation:

$$
2\partial_{\tau\theta}E_{x_0} = -\partial_{\tau\tau}E_{x_0} + \partial_{\theta\theta}E_{x_0}^3. \tag{526}
$$

Without the $\partial_{\tau\tau}$ will equation (526) become

$$
\begin{aligned}
2\partial_{\tau\theta}E_{x_0} &= \partial_{\theta\theta}E_{x_0}^3, && (527)\\
\implies 2\partial_\tau E_{x_0} &= \partial_\theta E_{x_0}^3 + O(\epsilon^2), && (528)\\
& && (529)
\end{aligned}
$$

The reason why our iteration procedure is expected to work is because $\partial_{\tau\tau}E_{x_0}$ is a small correction in (526). From (528) we get by taking the derivative with respect to τ

$$2\partial_{\tau\tau}E_{x_0} = \partial_\tau(\partial_\theta E_{x_0}^3), \tag{530}$$
$$= \partial_\theta(\partial_\tau E_{x_0}^3), \tag{531}$$
$$= \partial_\theta(3E_{x_0}^2\partial_\tau E_{x_0}), \tag{532}$$

and inserting (528) will make the equation

$$2\partial_{\tau\tau}E_{x_0} = \partial_\theta\left(\frac{1}{2}E_{x_0}^2\partial_\theta E_{x_0}^3\right), \tag{533}$$
$$= \partial_\theta\left(\frac{9}{2}E_{x_0}^4\partial_\theta E_{x_0}\right). \tag{534}$$

Inserting (534) into equation (526) will give

$$2\partial_{\tau\theta}E_{x_0} = -\partial_\theta\left(\frac{9}{4}E_{x_0}^4\partial_\theta E_{x_0}\right) + \partial_{\theta\theta}E_{x_0}^3. \tag{535}$$

This will give the equation

$$2\partial_\tau E_{x_0} = -\frac{9}{4}E_{x_0}^4\partial_\theta E_{x_0} + \partial_\theta E_{x_0}^3 + f(\tau), \tag{536}$$

We disregard $f(\tau)$, and end up with the equation

$$\implies \partial_\tau E_{x_0} = \frac{3}{2}E_{x_0}^2\left(1 - \frac{3}{4}E_{x_0}^2\right)\partial_\theta E_{x_0}. \tag{537}$$

This is also a quasilinear first order partial differential equation with the initial value $E_{x_0}(\theta, 0) = f(\theta)$. It can therefore be solved using the method of characteristics [6]. First parameterize the initial curve:

$$\theta = t \qquad \tau = 0 \qquad E_{x_0} = f(t), \tag{538}$$

and find the value of

$$J = \frac{\partial\tau}{\partial t}(-\frac{3}{2}E_{x_0}^2(1 - \frac{3}{4}E_{x_0}^2)) - \frac{\partial\theta}{\partial t}(1) = 0 - 1 = -1 \neq 0. \tag{539}$$

This means that there exists one and only one solution to this equation. The characteristic equations are

$$\frac{\partial \theta}{\partial s} = -\frac{3}{2}E_{x_0}^2(1-\frac{3}{4}E_{x_0}^2) \qquad \frac{\partial \tau}{\partial s} = 1 \qquad \frac{\partial E_{x_0}}{\partial s} = 0, \tag{540}$$

with the initial condition $s = 0$.

Here will

$$\frac{\partial \theta}{\partial s} = -\frac{3}{2}E_{x_0}^2(1-\frac{3}{4}E_{x_0}^2)$$

become

$$\theta = -\frac{3}{2}E_{x_0}^2\left(1-\frac{3}{4}E_{x_0}^2\right)s + t. \tag{541}$$

$$\frac{\partial \tau}{\partial s} = 1$$

will become

$$\tau = s + c. \tag{542}$$

Because of the initial condition $s = 0$ will $c = 0$ and

$$\tau = s. \tag{543}$$

$$\frac{\partial E_{x_0}}{\partial s} = 0 \tag{544}$$

means that $E_{x_0}(s,t)$ is constant along the characteristic curves such that

$$E_{x_0}(s,t) = E_{x_0}(0,t) = f(t). \tag{545}$$

Inserting (541) and (543) will give the equation

$$\theta = -\frac{3}{2}f(t)^2\left(1-\frac{3}{4}f(t)^2\right)\tau + t, \tag{546}$$

$$\implies t = \theta + \frac{3}{2}f(t)^2\left(1-\frac{3}{4}f(t)^2\right)\tau, \tag{547}$$

$$= \theta + \frac{3}{2}E_{x_0}^2\left(1-\frac{3}{4}E_{x_0}^2\right)\tau. \tag{548}$$

This will give the implicit solution

$$E_{x_0} = f(\theta + \frac{3}{2}E_{x_0}^2\left(1-\frac{3}{4}E_{x_0}^2\right)\tau). \tag{549}$$

6.1 Breaking Time

Start with equation (549) :

$$E_{x_0} = f(\theta + \frac{3}{2}E_{x_0}^2\left(1 - \frac{3}{4}E_{x_0}^2\right)\tau).$$

$$\partial_\theta E_{x_0} - 3E_{x_0}\tau\partial_\theta E_{x_0} f'(t) + \frac{9}{2}E_{x_0}^3\partial_\theta E_{x_0} f'(t) = f'(t), \tag{550}$$

$$\partial_\theta E_{x_0}\left(1 - 3E_{x_0}\tau f'(t) + \frac{9}{2}E_{x_0}^3\tau f'(t)\right) = f'(t), \tag{551}$$

$$\partial_\theta E_{x_0} = f'\left(\theta + \frac{3}{2}E_{x_0}^2\tau - \frac{9}{8}E_{x_0}^4\tau\right)\left(1 + \frac{3}{2}2E_{x_0}\tau\partial_\theta E_{x_0} - \frac{9}{8}4E_{x_0}^3\tau\partial_\theta E_{x_0}\right), \tag{552}$$

$$= f'\left(\theta + \frac{3}{2}E_{x_0}^2\tau - \frac{9}{8}E_{x_0}^4\tau\right) + 3E_{x_0}\tau\partial_\theta E_{x_0} f'(t) - \frac{9}{2}E_{x_0}^3\tau\partial_\theta E_{x_0} f'(t), \tag{553}$$

$$\implies \partial E_{x_0} = \frac{f'(t)}{1 - 3f(t)f'(t)\tau + \frac{9}{2}f(t)^3 f'(t)\tau}. \tag{554}$$

Breakdown when

$$f'(t)1 - 3f(t)f'(t)\tau + \frac{9}{2}f(t)^3 f'(t)\tau = 0, \tag{555}$$

$$\implies \tau\left(3f(t)f'(t) - \frac{9}{2}f(t)^3 f'(t)\right) = 1, \tag{556}$$

$$\implies \tau = \frac{1}{3f(t)f'(t) - \frac{9}{2}f(t)^3 f'(t)}, \tag{557}$$

so it's breakdown time when

$$\tau* = \min_t\left(\frac{1}{3f(t)f'(t) - \frac{9}{2}f(t)^3 f'(t)}\right). \tag{558}$$

6.2 Numerical Solution

The finite difference method [2] is a numerical method that is used to find the numerical solution of ordinary differential equation. The method solves the ordinary differential equation by first discretization of the equations on a space- time grid, that is shown in figure 3.

The first line is given. Center difference uses points from the two prior lines to find the next point, and therefore is it impossible to use center difference to find the second line. That's why we have to use forward difference for the derivative on τ. This makes the equation for the second line on the space-time grid (without the boundary points)

$$(E_{x_0})_i^1 = (E_{x_0})_i^0 + \frac{3}{4}s((E_{x_0})_i^0)^2((E_{x_0})_{i+1}^0 - (E_{x_0})_{i-1}^)(1 - \frac{3}{4}((E_{x_0})_i^0)^2), \tag{559}$$

where $s = \frac{\partial\tau}{\partial\theta}$, which it is all the time.

Because of the boundary conditions will the equation for the boundary points need to be different from (559) with the derivative on θ. So the equation for the second line and i=0 is

$$(E_{x_0})_0^1 = (E_{x_0})_0^0 + \frac{3}{4}s((E_{x_0})_0^0)^2((E_{x_0})_1^0 - (E_{x_0})_{N-2}^0)(1 - \frac{3}{4}((E_{x_0})_0^0)^2). \tag{560}$$

And the equation for the second line and i = N-2

$$(E_{x_0})_{N-2}^1 = (E_{x_0})_{N-2}^0 + \frac{3}{4}s((E_{x_0})_{N-2}^0)^2((E_{x_0})_0^0 - (E_{x_0})_{N-3}^0)(1 - \frac{3}{4}((E_{x_0})_{N-2}^0)^2). \tag{561}$$

Now do we have the second equation. Then is it possible to use central difference for the rest of the lines, so the equation for the rest of the lines (except from the boundary points) is

$$(E_{x_0})_i^{n+1} = (E_{x_0})_i^{n-1} + \frac{3}{2}s((E_{x_0})_i^n)^2((E_{x_0})_{i+1}^n - (E_{x_0})_{i-1}^n)(1 - \frac{3}{4}((E_{x_0})_i^n)^2). \tag{562}$$

The boundary points still need their own equations because of the boundary conditions. The center difference equation for the boundary point i=0 is

$$(E_{x_0})_0^{n+1} = (E_{x_0})_0^{n-1} + \frac{3}{2}s((E_{x_0})_0^n)^2((E_{x_0})_1^n - (E_{x_0})_{N-3}^n)(1 - \frac{3}{4}((E_{x_0})_{N-2}^n)^2), \tag{563}$$

. and the center difference boundary point i=N-2 is

$$(E_{x_0})_{N-2}^{n+1} = (E_{x_0})_{N-2}^{n-1} + \frac{3}{2}s((E_{x_0})_{N-2}^n)^2((E_{x_0})_0^n - (E_{x_0})_{N-3}^n)(1 - \frac{3}{4}((E_{x_0})_{N-2}^n)^2). \tag{564}$$

6.2.1 Stability

The numerical solution can have an instability. This can be avoided by finding a stability condition.

To find the stability condition let's look at the linear case of equation (537)

$$\partial_\tau E_{x_0} + c\partial_\theta E_{x_0} = 0, \tag{565}$$

where

$$c = c(E_{x_0}) = \frac{3}{2}E_{x_0}^2\left(1 - \frac{3}{4}E_{x_0}^2\right). \tag{566}$$

Equation (565) is now the same as (496), and the stability analysis will be the same as for equations (526) to (512). There do we find out that finite difference method is stable if $s < 1$ for the linear case. In the quasilinear case of equation (537) will we have

$$s < \frac{1}{\max c(E_{x_0})} = \frac{1}{\max\left(\frac{3}{2}E_{x_0}^2\left(1 - \frac{3}{4}E_{x_0}^2\right)\right)} \tag{567}$$

6.2.2 Testing

Choosing the Gaussian function as a starting function to the numerical solution

$$f(x) = \exp(-\gamma x^2). \tag{568}$$

The derivative is

$$f'(x) = -2\gamma x \exp(-\gamma x^2). \tag{569}$$

Inserting (568) and (569) into equation (558) give us the equation $g(x)$,

$$g(x) = \frac{\exp(2\gamma x^2)}{9\gamma x \exp(-2\gamma x^2) - 6\gamma x}. \tag{570}$$

The value of x where $g(x)$ has it's minimum value is found by solving the equation where

$$g'(x) = 0. \tag{571}$$

Equation (571) become

$$\exp(2\gamma x^2) = \frac{72\gamma^2 x^2 - 9\gamma}{24\gamma^2 x^2 - 6\gamma}. \tag{572}$$

Now we are using Mathematica to find the solution (572). This is done by using the Mathematica function *FindRoot*. But this function needs an approximate value to start from when looking for the root. To do this we simply plot the right side and the left side of equation (572)to see where they intersect. To be able to do a plot we need to choose a value for γ, and in this case we choose

$$\gamma = 0.001. \tag{573}$$

By trial and error we find that the value of x that will give the smallest value for $\tau*$ lies somewhere inside the domain $x \in [-30, -22]$, as shown in figure 11. In this figure we can see that the intersection point is close to $x = -26$. Using $x = -26$ as the starting point we find the value of x where $g(x)$ has its minimum value is $x = -26.03$. Inserting this into equation (558) give the breaking point

$$\tau* = 40.49 \tag{574}$$

Comparing this to the numerical breaking time will then test that the numerical implementation is correct. Figure 12 shows the wave at time $\tau = 40.5$. The vertical line at the left shows that the graph breaks at this time, which means that the numerical implementation is correct.

6.3 Results

When $\gamma = 0.01$ the initial wave will look the same as in figure 7. And in the same way as in figure 8 the wave will start to lean to the left. For this case is the movement of the wave before breaking point shown in figure 13.

The breaking time for this case is

$$\tau* = 1.27, \tag{575}$$

which is shown in figure 14. The vertical line in the graph of figure 14 show that there is no longer possible to find a valid solution to equation (537).After the breaking point the wave will continue to get worse, as shown in equation 15.

7 Perturbation equations to order ϵ^2 with polarization

Starting again with equations (449) and (450):

$$\begin{aligned}2\partial_{\tau\theta}E_{x_0} = &-\partial_{yy}E_{x_0} - \partial_{xx}E_{x_0} - \partial_{\tau\tau}E_{x_0}\\ &- (\epsilon^2\partial_{xx}(\hat{\chi}(-ic\partial_\theta)E_{x_0} + \eta(E_{x_0}^3 + E_{y_0}^2E_{x_0}))\\ &+ \epsilon^2\partial_{xy}(\epsilon_0\hat{\chi}(-ic\partial_\theta)E_{y_0} + \epsilon_0\eta(E_{x_0}^2E_{y_0} + E_{y_0}^3))+\\ &\epsilon^3\partial_{x\theta}\epsilon_0\eta(E_{x_0}^2 + E_{y_0}^2)E_{z_1})\\ &+ \mu_0\epsilon^2c^2\partial_{\theta\theta}(\epsilon_0\hat{\chi}(-ic\partial_\theta)E_{x_0} + \epsilon_0\eta(E_{x_0}^3 + E_{y_0}^2E_{x_0})),\end{aligned}$$

$$\begin{aligned}2\partial_{\tau\theta}E_{y_0} = &-\partial_{yy}E_{y_0} - \partial_{xx}E_{y_0} - \partial_{\tau\tau}E_{y_0}\\ &- (\epsilon^2\partial_{xy}(\hat{\chi}(-ic\partial_\theta)E_{x_0} + \eta(E_{x_0}^3 + E_{y_0}^2E_{x_0}))+\\ &\epsilon^2\partial_{yy}(\epsilon_0\hat{\chi}(-ic\partial_\theta)E_{y_0} + \eta\epsilon_0(E_{x_0}^2E_{y_0} + E_{y_0}^3))\\ &+ \epsilon^3\partial_{y\theta}\epsilon_0\eta(E_{x_0}^2 + E_{y_0}^2)E_{z_1}) + \mu_0\epsilon_0c^2\epsilon^2\partial_{\theta\theta}(\hat{\chi}(-ic\partial_\theta)E_{y_0}\\ &+ \eta(E_{x_0}^2E_{y_0} + E_{y_0}^3)).\end{aligned}$$

Removing the dispersion and the diffraction and retaining only terms at order ϵ^2 will give the equations:

$$2\partial_{\theta\tau}E_x = \mu_0\epsilon_0c^2\epsilon^2\partial_{\theta\theta}\eta(E_{x_0}^3 + E_{x_0}E_{y_0}^2), \tag{576}$$
$$2\partial_{\theta\tau}E_y = \mu_0\epsilon_0c^2\epsilon^2\partial_{\theta\theta}\eta(E_{x_0}^2E_{y_0} + E_{y_0}^3). \tag{577}$$

Integrating over θ on both sides will give:

$$2\partial_\tau E_x = \mu_0\epsilon_0c^2\epsilon^2\partial_\theta\eta(E_{x_0}^3 + E_{x_0}E_{y_0}^2) + f(\tau), \tag{578}$$
$$2\partial_\tau E_y = \mu_0c^2\epsilon_0\epsilon^2\partial_\theta\eta(E_{x_0}^2E_{y_0} + E_{y_0}^3) + g(\tau). \tag{579}$$

This time are we going to disregard $f(\tau)$ and $g(\tau)$. Saying that $\mu_0\epsilon_0\eta c^2\epsilon^2 = \alpha$:

$$2\partial_\tau E_x = \alpha\partial_\theta(E_{x_0}^3 + E_{x_0}E_{y_0}^2), \tag{580}$$
$$2\partial_\tau E_y = \alpha\partial_\theta(E_{x_0}^2E_{y_0} + E_{y_0}^3). \tag{581}$$

This time will $E_{y_0} = 0$, so we're ending up with a system of equations. Doing a change of variables

$$\theta = \theta_0 \theta', \tag{582}$$
$$\tau = \tau_0 \tau'. \tag{583}$$

The chain rule will then give

$$\frac{\partial}{\partial \theta} = \frac{1}{\theta_0} \frac{\partial}{\partial \theta'}, \tag{584}$$
$$\frac{\partial}{\partial \tau} = \frac{1}{\tau_0} \frac{\partial}{\partial \tau'}. \tag{585}$$

Inserting (584) and (585) into equations (580) and (581) will give

$$2 \frac{1}{\theta_0 \tau_0} \frac{\partial}{\partial \theta' \partial \tau'} E_{x_0} = \alpha \frac{1}{\theta_0^2} \frac{\partial}{\partial \theta' \partial \theta'} (E_{x_0}^3 + E_{x_0} E_{y_0}^2), \tag{586}$$
$$2 \frac{1}{\theta_0 \tau_0} \frac{\partial}{\partial \theta' \partial \tau'} E_{y_0} = \alpha \frac{1}{\theta_0^2} \frac{\partial}{\partial \theta' \partial \theta'} (E_{y_0}^3 + E_{y_0} E_{x_0}^2). \tag{587}$$

Multiplying equation (586) and (587) by $\tau_0 \theta_0$ will give the equations

$$2 \partial_{\tau'} E_{x_0} = \frac{\alpha \tau_0}{\theta_0} \partial_{\theta'} (E_{x_0}^3 + E_{x_0} E_{y_0}^2), \tag{588}$$
$$2 \partial_{\tau'} E_{y_0} = \frac{\alpha \tau_0}{\theta_0} \partial_{\theta'} (E_{y_0}^3 + E_{y_0} E_{x_0}^2). \tag{589}$$

And then choosing

$$\frac{\alpha \tau_0}{\theta_0} = 1$$

will give the system of equations

$$2 \partial_{\tau'} E_{x_0} = \partial_{\theta'} (E_{x_0}^3 + E_{x_0} E_{y_0}^2), \tag{590}$$
$$2 \partial_{\tau'} E_{y_0} = \partial_{\theta'} (E_{y_0}^3 + E_{y_0} E_{x_0}^2). \tag{591}$$

Equation (590) becomes

$$2 \partial_{\tau'} E_{x_0} = \partial_{\theta'} E_{x_0}^3 + \partial_{\theta} (E_{x_0} E_{y_0}^2)), \tag{592}$$
$$\implies 2 \partial_{\tau'} E_{x_0} = 3 E_{x_0} \partial_{\theta'} E_{x_0} + E_{y_0}^2 \partial_{\theta'} E_{x_0} + 2 E_{x_0} E_{y_0} \partial_{\theta'} E_{y_0}, \tag{593}$$
$$\implies 2 \partial_{\tau'} E_{x_0} = (3 E_{x_0}^2 + E_{y_0}^2) \partial_{\theta'} E_{x_0} + 2 E_{x_0} E_{y_0} \partial_{\theta'} E_{y_0}. \tag{594}$$

And equation (591) becomes

$$2\partial_{\tau'}E_{y_0} = \partial_{\theta'}E_{y_0}^3 + \partial_{\theta}(E_{y_0}E_{x_0}^2)), \tag{595}$$

$$\implies 2\partial_{\tau'}E_{y_0} = 3E_{y_0}\partial_{\theta'}E_{y_0} + E_{x_0}^2\partial_{\theta'}E_{y_0} + 2E_{y_0}E_{x_0}\partial_{\theta'}E_{x_0}, \tag{596}$$

$$\implies 2\partial_{\tau'}E_{y_0} = (3E_{y_0}^2 + E_{x_0}^2)\partial_{\theta'}E_{y_0} + 2E_{y_0}E_{x_0}\partial_{\theta'}E_{x_0}. \tag{597}$$

7.1 Numerical Solution

The finite difference method [2] solves the ordinary differential equation by first discretization of the equations on the space- time grid shown in figure 3. That means that the equation $E_{x_0}(\theta, \tau)$ becomes $E_{x_0}(\theta_i, \tau_n)$.

The two equations will be solved by the same way as before, where they are solved separately by finite differences. It's just important to find all the points in each variable before moving on to the points in the next line. The first line for each variable is given. Center difference uses points from the two prior lines to find the next point, and therefore is it impossible to use center difference to find the second lines. That's why it's necessary to use forward difference on the derivative of τ. This makes the equations for the second lines on the space- time grids (without the boundary points). First for (594):

$$\begin{aligned}(E_{x_0})_i^1 = (E_{x_0})_i^0 + \frac{1}{4}s(3((E_{x_0})_i^0)^2 + ((E_{y_0})_i^0)^2)((E_{x_0})_{i+1}^0 - (E_{x_0})_{i-1}^0) \\ + \frac{1}{2}s(E_{x_0})_i^0(E_{y_0})_i^0((E_{y_0})_{i+1}^0 - (E_{y_0})_{i-1}^0),\end{aligned} \tag{598}$$

and for equation (597)

$$\begin{aligned}(E_{y_0})_i^1 = (E_{y_0})_i^0 + \frac{1}{4}s(3((E_{y_0})_i^0)^2 + ((E_{x_0})_i^0)^2)((E_{y_0})_{i+1}^0 - (E_{y_0})_{i-1}^0) \\ + \frac{1}{2}s(E_{x_0})_i^0(E_{y_0})_i^0((E_{x_0})_{i+1}^0 - (E_{x_0})_{i-1}^0).\end{aligned} \tag{599}$$

where $s = \frac{\partial\tau}{\partial\theta}$, which it is always. Because of the boundary conditions will the equations for the boundary points need to be different from (598) and (599). First for the boundary points i=0 on the second line. First for equation (594)

$$\begin{aligned}(E_{x_0})_0^1 = (E_{x_0})_0^0 + \frac{1}{4}s(3((E_{x_0})_0^0)^2 + ((E_{y_0})_0^0)^2)((E_{x_0})_1^0 - (E_{x_0})_{N-2}^0) \\ + \frac{1}{2}s(E_{x_0})_0^0(E_{y_0})_0^0((E_{y_0})_1^0 - (E_{y_0})_{N-2}^0),\end{aligned} \tag{600}$$

and for equation (597)

$$(E_{y_0})_0^1 = (E_{y_0})_0^0 + \frac{1}{4}s(3((E_{y_0})_0^0)^2 + ((E_{x_0})_0^0)^2)((E_{y_0})_1^0 - (E_{y_0})_{N-2}^0) + \frac{1}{2}s(E_{x_0})_0^0(E_{y_0})_0^0((E_{x_0})_1^0 - (E_{x_0})_{N-1}). \tag{601}$$

And then for the boundary point i=N-2 for the second line. For equation (594) is it

$$(E_{x_0})_{N-2}^1 = (E_{x_0})_{N-2}^0 + \frac{1}{4}s(3((E_{x_0})_{N-2}^0)^2 + ((E_{y_0})_{N-2}^0)^2)((E_{x_0})_0^0 - (E_{x_0})_{N-3}^0) + \frac{1}{2}s(E_{x_0})_{N-2}^0(E_{y_0})_{N-2}^0((E_{y_0})_0^0 - (E_{y_0})_{N-3}^0), \tag{602}$$

and for equation (597) is it

$$(E_{y_0})_{N-2}^0 = (E_{y_0})_{N-2}^0 + \frac{1}{4}s(3((E_{x_0})_{N-2}^0)^2 + ((E_{y_0})_{N-2}^0)^2)((E_{y_0})_0^0 - (E_{y_0})_{N-3}^0) + \frac{1}{2}s(E_{x_0})_{N-2}^0(E_{y_0})_{N-2}^0((E_{x_0})_0^0 - (E_{x_0})_{N-3}^0). \tag{603}$$

Now do we have the whole second line, and it is then possible to use center difference to find the general equation for the rest of the lines on the space-time grids (except of the boundary points). Starting with equation (594)

$$(E_{x_0})_i^{n+1} = (E_{x_0})_i^{n-1} + \frac{1}{2}s(3((E_{x_0})_i^n)^2 + ((E_{y_0})_i^n)^2)((E_{x_0})_{i+1}^n - (E_{x_0})_{i-1}^n) + s(E_{x_0})_i^n(E_{y_0})_i^n((E_{y_0})_{i+1}^n - (E_{y_0})_{i-1}^n), \tag{604}$$

and for equation (597):

$$(E_{y_0})_i^{n+1} = (E_{y_0})_i^{n-1} + \frac{1}{2}s(3((E_{y_0})_i^n)^2 + ((E_{x_0})_i^n)^2)((E_{y_0})_{i+1}^n - (E_{y_0})_{i-1}^n) + s(E_{y_0})_i^n(E_{x_0})_i^n((E_{x_0})_{i+1}^n - (E_{x_0})_{i-1}^n). \tag{605}$$

The general center difference equation for the boundary point i=0 for equation (594)

$$(E_{x_0})_0^{n+1} = (E_{x_0})_0^{n-1} + \frac{1}{2}s(3((E_{x_0})_0^n)^2 + ((E_{y_0})_0^n)^2)((E_{x_0})_1^n - (E_{x_0})_{N-2}^n) + s(E_{x_0})_0^n(E_{y_0})_0^n((E_{y_0})_1^n - (E_{y_0})_{N-2}^n), \tag{606}$$

and the equation for the boundary point i=0 for equation (597)

$$(E_{y_0})_0^{n+1} = (E_{y_0})_0^{n-1} + \frac{1}{2}s(3((E_{y_0})_0^n)^2 + ((E_{x_0})_0^n)^2)((E_{y_0})_1^n - (E_{y_0})_{N-2}^n) + s(E_{x_0})_0^n(E_{y_0})_0^n((E_{x_0})_1^n - (E_{x_0})_{N-2}^n). \tag{607}$$

The general center difference equation for the boundary point i=N-2 for equation (594)

$$(E_{x_0})_{N-2}^{n+1} = (E_{x_0})_{N-2}^{n-1} + \frac{1}{2}s(3((E_{x_0})_{N-2}^n)^2 + ((E_{y_0})_{N-2}^n)^2)((E_{x_0})_0^n - (E_{x_0})_{N-3}^n) + s(E_{x_0})_{N-2}^n(E_{y_0})_{N-2}^n((E_{y_0})_0^n - (E_{y_0})_{N-3}^n), \tag{608}$$

and the equation for the boundary point i= N-2 for equation (597) is

$$(E_{y_0})_{N-2}^{n+1} = (E_{y_0})_{N-2}^{n-1} + \frac{1}{2}s(3((E_{y_0})_{N-2}^n)^2 + ((E_{x_0})_{N-2}^n)^2)((E_{y_0})_0^n - (E_{y_0})_{N-3}^n) + s(E_{x_0})_{N-2}^n(E_{y_0})_{N-2}^n((E_{x_0})_0^n - (E_{y_0})_{N-3}^n). \tag{609}$$

7.1.1 Initial functions

The initial incoming laser pulse is a polarized wave packet of the form

$$E_{x_0} = f(t)\cos(\omega_0 t), \tag{610}$$
$$E_{y_0} = f(t)\cos(\omega_0 t + \varphi), \tag{611}$$

where

$$0 \leq \varphi \leq 2\pi. \tag{612}$$

The variable φ takes into account polarization. For the case when $f(t) = const$, the components e_x and e_y will be periodic and define an ellipse in the (e_x, e_y) plane, as the example of figure 16. Using the change of variables (37)-(38) to equation (610)-(611) give the equations

$$A(\theta, 0) = f\left(-\frac{\theta}{c}\right)\cos\left(-\omega_0\frac{\theta}{c}\right), \tag{613}$$
$$B(\theta, 0) = f\left(-\frac{\theta}{c}\right)\cos\left(-\omega_0\frac{\theta}{c} + \varphi\right). \tag{614}$$

We will use a Gaussian envelope equation that is symmetric around t=0.

$$f(t) = a\exp(-bt^2). \tag{615}$$

Introducing the scales

$$\tau = \tau_0\tau', \tag{616}$$
$$\theta = \theta_0\theta', \tag{617}$$
$$A = \alpha_0 A', \tag{618}$$
$$B = \alpha_0 B', \tag{619}$$

will give the equations

$$A'(\theta', 0) = \frac{a}{\alpha_0}\exp\left(-\frac{b}{c^2}\theta_0^2\theta'^2\right)\cos\left(\frac{\omega_0\theta_0}{c}\theta'\right), \tag{620}$$
$$B'(\theta', 0) = \frac{a}{\alpha_0}\exp\left(-\frac{b}{c^2}\theta_0^2\theta'^2\right)\cos\left(\frac{\omega_0\theta_0}{c}\theta' + \varphi\right). \tag{621}$$

Choosing $\alpha = a$ will normalize the amplitude to one, and choosing the scaling θ_0 in such a way that the carrier wave is of period 2π gives

$$\frac{\omega_0\theta_0}{c} = 1, \tag{622}$$
$$\theta_0 = \frac{c}{\omega_0}. \tag{623}$$

These choices give us the equations

$$A'(\theta', 0) = \exp(-\gamma\theta'^2)\cos(\theta'), \tag{624}$$
$$B'(\theta', 0) = \exp(-\gamma\theta'^2)\cos(\theta' + \varphi), \tag{625}$$

where $\gamma = \frac{b}{\omega_0^2}$. Since b is free can γ be a free dimensionless number.

7.1.2 Stability

Equations (594) and (597) are a system of equations on the form

$$\partial_t U = a\partial_x U + b\partial_x V, \tag{626}$$
$$\partial_t V = a\partial_x v + b\partial_x U. \tag{627}$$

Equation (626) and (627) can be written on the form

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \partial_x \begin{pmatrix} U \\ V \end{pmatrix}, \tag{628}$$

where matrix A is

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

The eigenvalues of our system will be

$$det \begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} = 0, \tag{629}$$

$$(\lambda - a)^2 - b^2 = 0, \tag{630}$$

$$\implies \lambda = a \pm b. \tag{631}$$

For a Gaussian choice for the starting function will the value of λ that means stability be the largest value $\lambda = a+b$. Because of the fact that in equations (594) and (597) is $a = \frac{3E_{x_0}^2 + E_{y_0}^2}{2}$ and $b = E_{x_0}E_{y_0}$ the stability condition will be

$$\lambda = a + b = \frac{3E_{x_0}^2 + E_{y_0}^2}{2} + E_{x_0}E_{y_0}. \tag{632}$$

Because we have chosen a Gaussian initial function with amplitude 1, will the stability condition be

$$\lambda = \frac{3+1}{2} + 1 = \frac{6}{2} = 3. \tag{633}$$

So the stability value for s is

$$\lambda s < 1, \tag{634}$$

$$s < \frac{1}{\lambda} \tag{635}$$

$$\implies s < \frac{1}{3}. \tag{636}$$

7.2 Results

For the case when $\gamma = 0.1$ will the starting function look like shown in figure 17. And the starting function when $\gamma = 0.01$ is as shown in figure 18.

The breaking times depending on the value of φ are shown in figure 18. The red line show the breaking time when $\gamma = 0.1$, and the blue line show the breaking times when $\gamma = 0.01$.

8 Discussion

In chapter 5 it is shown that the quasilinear equation will break at a time $\tau*$. The example shown in chapter 5.3 show that the equation will break at $\tau* = 0.59$ when $\gamma = 0.01$. When reintroducing $\partial_{\tau\tau}$ the equations will also break.

The example shown in chapter 6.3 is also when $\gamma = 0.01$, and that the breaking time then is $\tau* = 1.27$. This show that reintroducing $\partial_{\tau\tau}$ in the non dispersive, purely paraxial case of chapter 5 will not remove breakdown.

We can also see that reintroducing $\partial_{\tau\tau}$ to equation (458) will get the breakdown to come later than the breakdown in (458). However, the equation will still break at an early time. This means that the contribution $\partial_{\tau\tau}$ has to the breakdown is small.

In chapter 7 is it looked into how polarization influences the breaking time. In figure 19 is it shown that when γ gets smaller, the polarization will have a smaller effect on the breaking time. Remembering that $\gamma = \frac{b}{\omega_0^2}$, where b is a small dimensionless number.

This means that the dependence of the breaking time on the angle φ will increase when ω_0 increase for a fixed envelope shape. Thus the effect of polarization on breakdown increase when there are more carrier oscillations within the envelope.

9 Summary

We have been using the method of multiple scales to derive leading order asymptotic equations describing the propagation of approximately paraxial pulses in a weakly nonlinear and weakly dispersive media. When doing this we have first been looking at scalar equations, before moving on to vector equations, first to order ϵ^2 and then to order ϵ^4. This to build up the competence to be able to be able to do the final derivations.

After that we simplified the equations to look at the non-dispersive and purely paraxial case. These were investigated both of the leading order ϵ^2 and at order ϵ^4. The results there shows that in the case of ϵ^2, after a specific time our model will break down. Perturbing to the order ϵ^4 will make the breakdown time come later. But when perturbing to order ϵ^4 there will only be a small contribution, and the breakdown time only comes a little later.

In particular we have looked into how the polarization influences the shock. This is done in the case of leading order ϵ^2. The results from this shows that in that case the breaking time will depend more on polarization when there are more carrier oscillations within the envelope.

Further work that could be done after this is investigating through numerical simulations of the asymptotic equations in order to see how diffraction modify the optical shocks. It is also possible to find corrections to the leading order asymptotic equations by continuing the method of multiple scales to even higher order, and then investigating how these terms adding to the leading order equations modify the optical shocks in the non-dispersive purely paraxial case. Another thing that can be done further is to investigate how the presence of dispersion modify the optical shocks by numerical solution of the asymptotic equations, and use the Kerr coefficient and dispersion to for example the noble gas Xenon.

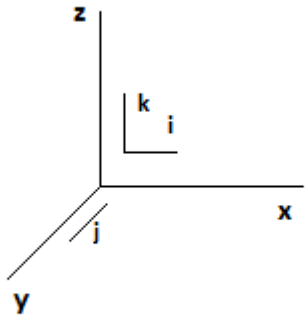

Figure 2: Coordinate system

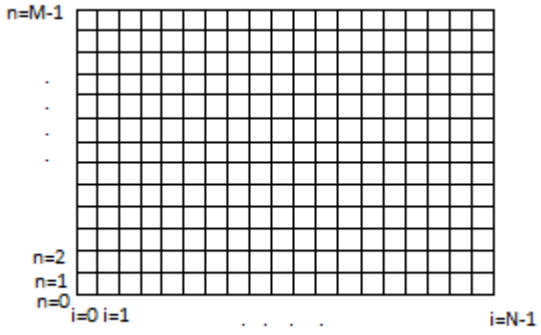

Figure 3: Space-time grid for (θ_i, τ_n), where i = 0,...,N-1, and n=0,...M-1.

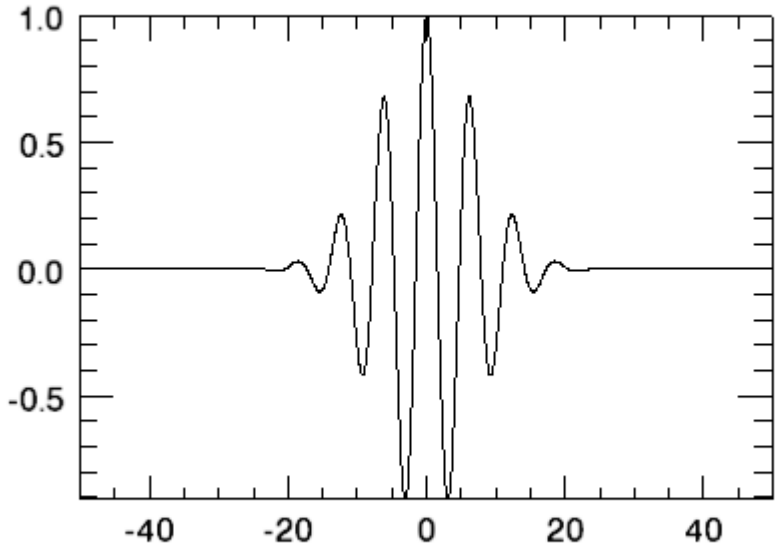

Figure 4: Shape of the initial laser pulse.

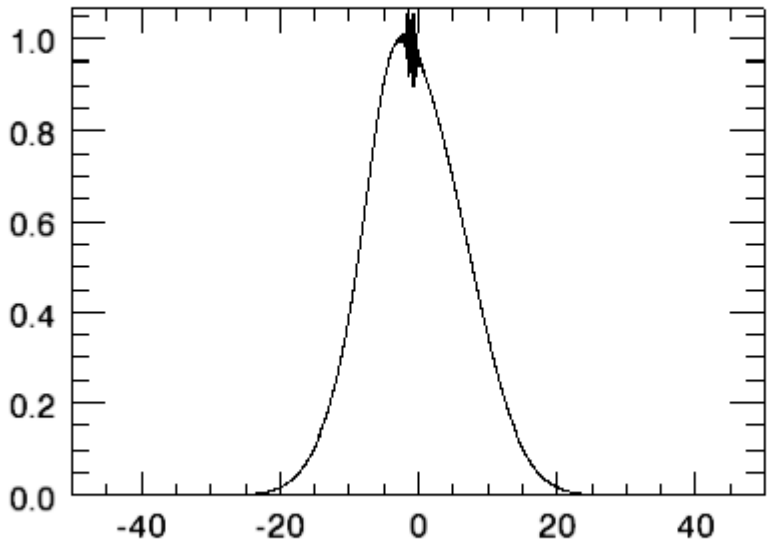

Figure 5: Graph that shows instability in the numerical method.

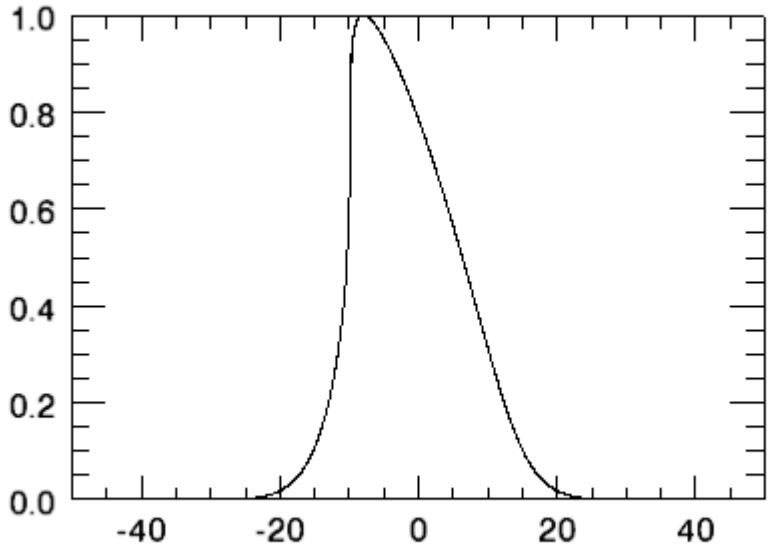

Figure 6: Graph for $\tau = 5.5$. The front of the graph is vertical, which means that the graph has broken.

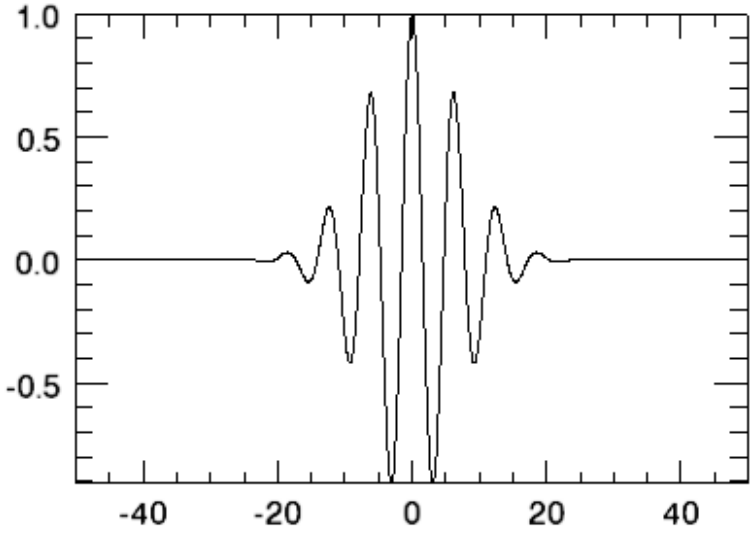

Figure 7: The initial incoming laser pulse

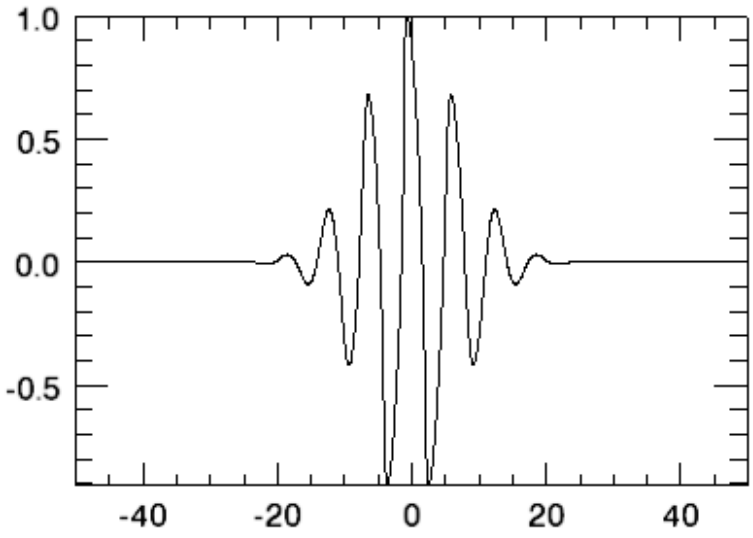

Figure 8: The nonlinearity of equation (458) makes the graph start to lean to the left.

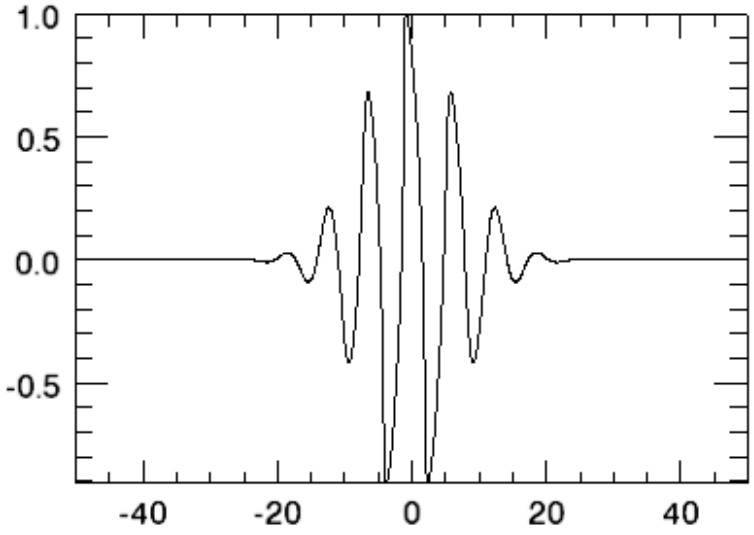

Figure 9: The graph in the breaking point. The vertical line shows that it has broken.

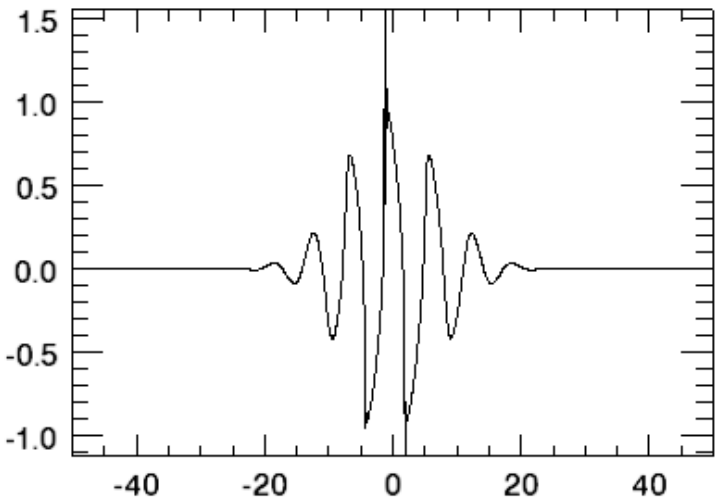

Figure 10: Graph of function after breaking point. This shows that it is now impossible to find a numerical solution.

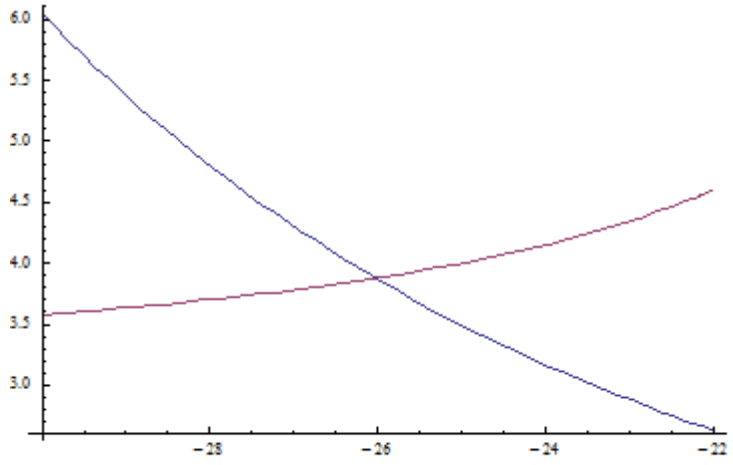

Figure 11: Graph of intersection of right and left side of (572) in the domain (-30,-22).

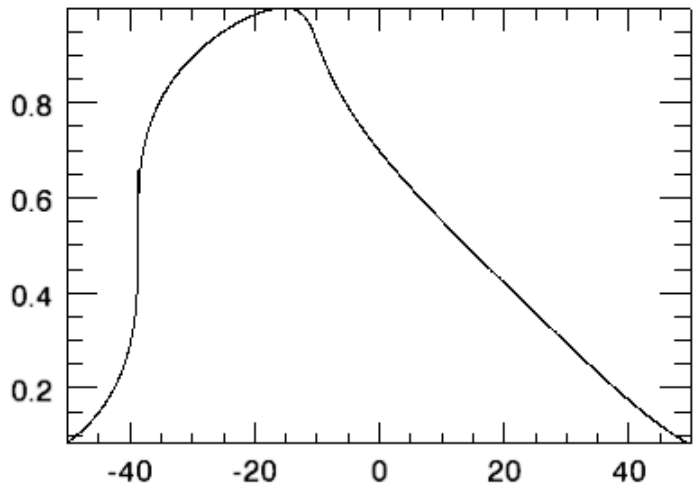

Figure 12: Graph of wave when $\tau = 40.5$.The front of the wave is vertical, which means that the wave has broken.

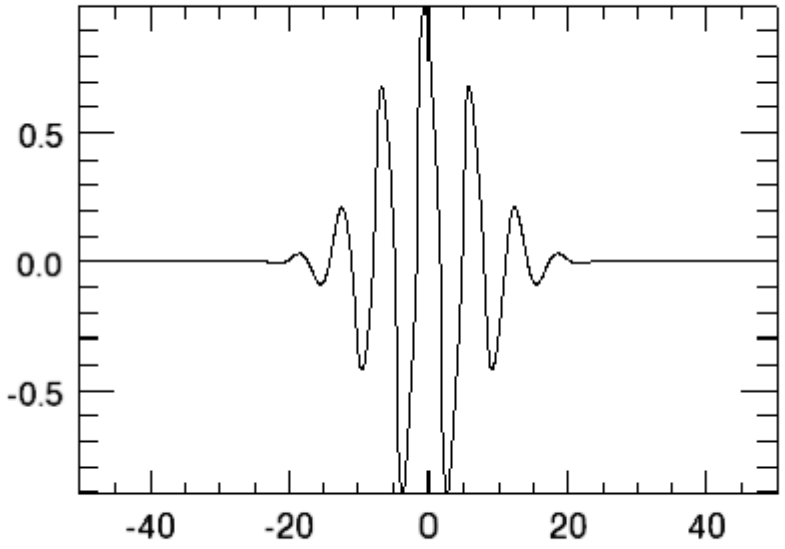

Figure 13: Graph for case 2 that shows how the wave moves before it reaches the breaking point.

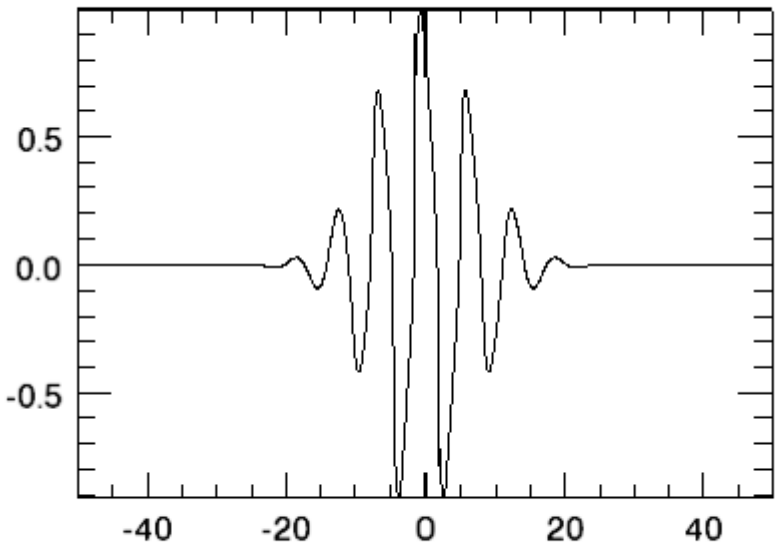

Figure 14: Graph that shows how the breaking point at case 2. The breaking point is shown by the vertical line in the graph.

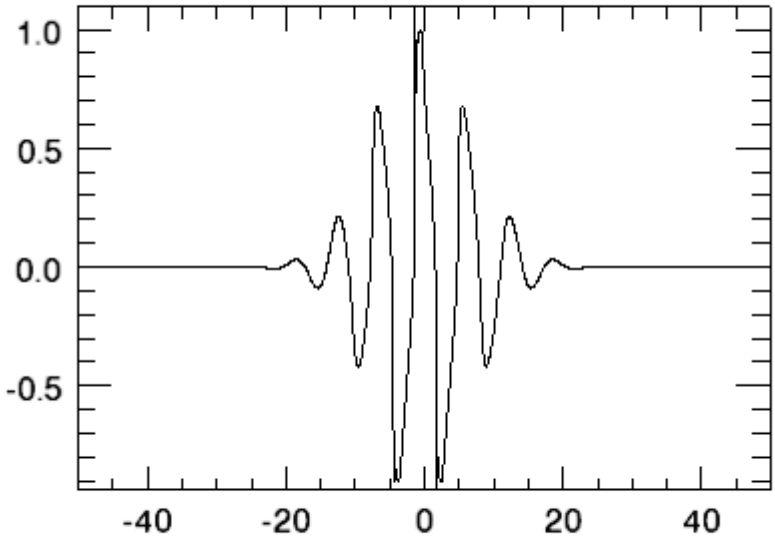

Figure 15: Graph at a time beyond the breaking point for case 2.

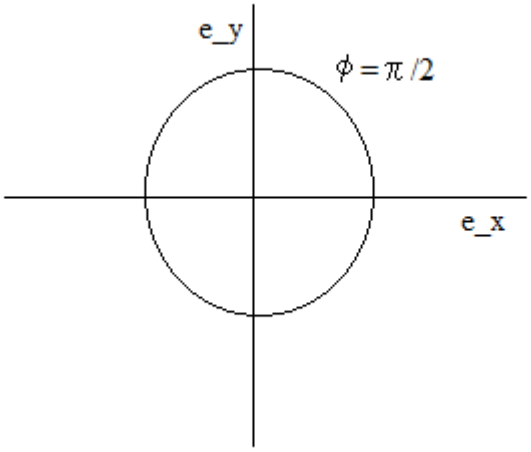

Figure 16: Example of an ellipse in the (e_x, e_y) plane

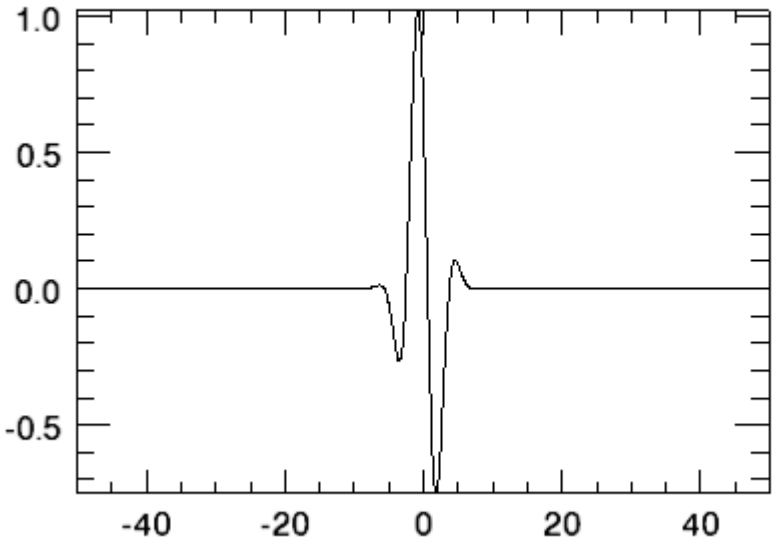

Figure 17: Initial wave packet when $\gamma = 0.1$.

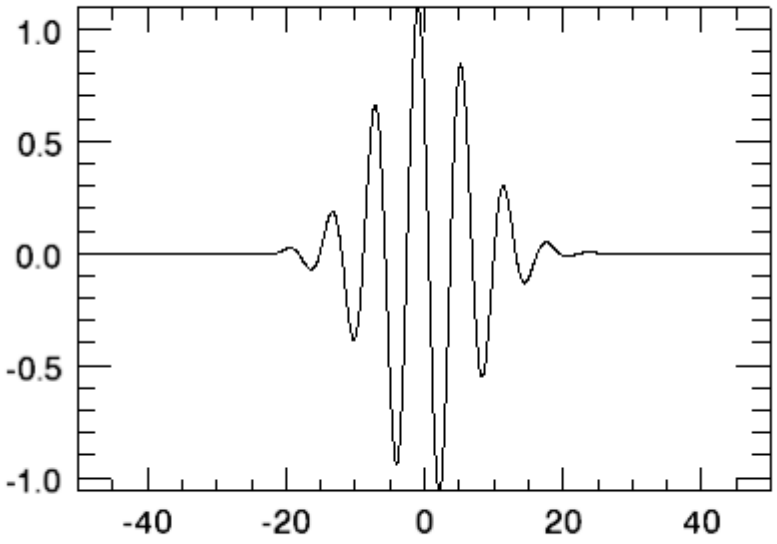

Figure 18: Initial wave packet when $\gamma = 0.01$.

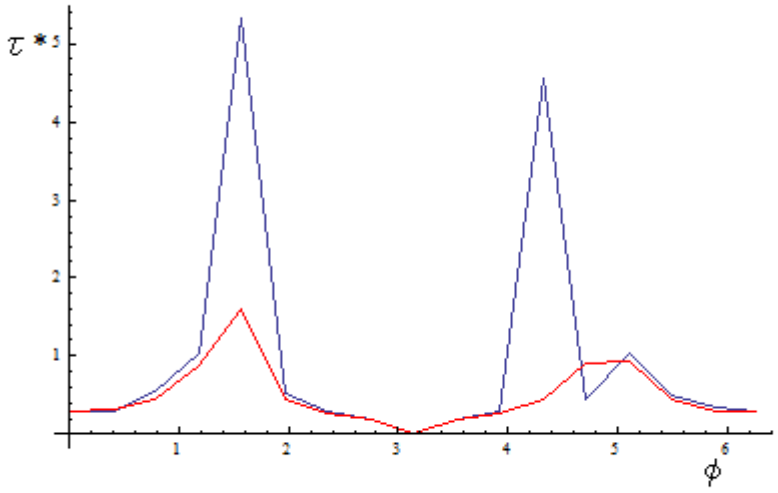

Figure 19: Figure showing the breaking times depending on the value of the polarization $\phi \in [0, 2\pi]$. The red line is when $\gamma = 0.1$, and the blue line is when $\gamma = 0.01$.

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